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# Analytical Dynamics and Nonrigid Spacecraft Simulation


*P. W. Likins*





(NASA-CR-139502) ANALYTICAL DYNAMICS AND NONRIGID SPACECRAFT SIMULATION (Jet Propulsion Lab.) N74-31333

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**JET PROPULSION LABORATORY**

**CALIFORNIA INSTITUTE OF TECHNOLOGY**

**PASADENA, CALIFORNIA**



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## Preface

The work described in this report was performed under the cognizance of the Guidance and Control Division of the Jet Propulsion Laboratory, which is supported by NASA contract NAS 7-100. The author is a Professor at UCLA and a consultant to JPL.

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# Abstract

This report contains an exposition of several alternative methods of analytical dynamics, and the application of these methods to alternative models of nonrigid spacecraft. This information permits the comparative evaluation of these methods for spacecraft simulation.

The following methods are developed from D'Alembert's principle in vector form:

(1) Lagrange's form of D'Alembert's principle for independent generalized coordinates.

(2) Lagrange's form of D'Alembert's principle for simply constrained systems.

(3) Kane's quasi-coordinate formulation of D'Alembert's principle.

(4) Lagrange's equations for independent generalized coordinates.

(5) Lagrange's equations for simply constrained systems.

(6) Lagrangian quasi-coordinate equations (or the Boltzmann–Hamel equations).

(7) Hamilton's equations for simply constrained systems.

(8) Hamilton's equations for independent generalized coordinates.

Applications to idealized spacecraft are considered both for multiple-rigid-body models and for models consisting of combinations of rigid bodies and elastic bodies, with the elastic bodies being defined either as continua, as finite-element systems, or as a collection of given modal data. Several specific examples are developed in detail by alternative methods of analytical mechanics, and results are compared to a Newton–Euler formulation.

Conclusions are straightforward in the case of the multiple-rigid-body topological tree idealization, for which the standard of comparison is a Newton–Euler formulation due originally to Hooker and Margulies and widely available in the form of a JPL computer program.

Although the equations in the previously existing JPL computer program are obtained in this report by means of both Kane's approach and the Lagrangian quasi-coordinate method, neither these nor any other methods of analytical dynamics produced results superior to the present standard.

Applications to combinations of rigid bodies and elastic bodies are more varied and more complex, and conclusions are more tentative, but essentially the same result emerges. Although various methods of analytical dynamics produce the *same* equations of motion as have previously been derived by the Newton–Euler approach, there appears to be no demonstrable advantage in any of the methods of analytical dynamics over the Newton–Euler results, except in the unusual case in which a continuum idealization is appropriate and in the somewhat academic case in which a truncated set of vibration mode shapes and frequencies are given in advance of the dynamic analysis.

# Analytical Dynamics and Nonrigid Spacecraft Simulation

## I. Introduction

### A. Background and Motivation

In the traditional academic perspective, the classical methods of Lagrange and Hamilton are, in comparison with the direct application of Newton's laws, accepted as the more advanced procedures for formulating equations of motion for mechanical systems.

The methods of Newton and Euler, which involve physically visualizable quantities represented in modern times by Gibbsian vectors[1] and dyadics, are generally recognized as being most useful in the struggle for conceptual understanding of the behavior of relatively simple systems, such as particles in space or gyroscopes. It is, however, widely believed that, in providing the transition from the physical world of *vectorial mechanics* to the abstract analytical realm of generalized scalar formulations found in *analytical mechanics*, Lagrange gave us superior procedures for deriving equations of motion for complex mechanical systems.

Hamilton's formulations are widely regarded as even more powerful than those of Lagrange. *Hamilton's principle* embraces much of Newtonian mechanics in a single, scalar variational equation; Hamilton's canonical equations replace Lagrange's scalar, second order, ordinary differential equations with first order ordinary differential equations of remarkably simple structure; and the Hamilton–Jacobi equation is a single partial differential equation that subsumes much of Newtonian and Lagrangian mechanics.

[1]A *Gibbsian vector* (to be distinguished from an $n$-dimensional column matrix) is geometrically equivalent to a directed line segment in physical space, with rules for addition and both scalar and vectorial multiplication.

The methods of Lagrange and Hamilton automatically remove from the equations of motion most of the unknown and unwanted forces of constraint that plague the analyst who applies Newton's laws. Moreover, the former methods yield differential equations whose structure is system-invariant, while the procedures of Newton and Euler must be reconstructed for each new mechanical system. Finally, the equations of Lagrange and Hamilton are explicitly constructed to facilitate integration, whereas those of Newton and Euler have no particular structure at all, being dependent for their form on the strategy adopted by the analyst.

Against this background, we consider the notoriously complex problems of formulating equations of motion for nonrigid spacecraft. Probably no other class of physical system is routinely subjected to such complicated mathematical modeling, and described by such difficult ordinary differential equations of motion. Fortunately, the spacecraft and its physical environment are much more *amenable* to accurate modeling than are other physical systems of comparable or greater complexity (such as the automobile, or the human being). The internal structure of the spacecraft is subject to component testing and design control, and the external space environment is much less complex than the terrestrial environment. At the same time, spacecraft mission performance specifications demand extraordinary precision in the prediction of dynamic behavior; for example, certain planned astronomical observatory experiments require that a large space telescope maintain an established orientation to within an rms pointing accuracy measured in *thousandths of an arcsecond*! The combination of amenability to accurate modeling and demand for accurate predictions has resulted in great emphasis during the past five or ten years on the development of ever more complex formulations of the equations of motion of idealized space vehicles.

The juxtaposition of the modern spacecraft and the reputation of the methods of Lagrange and Hamilton certainly suggests that enlightened analysts would bring these methods to bear upon the spacecraft simulation problem. Yet the informed reader will recognize that at present the most widely used procedures for simulating nonrigid spacecraft are based on the methods of Newton and Euler. It is true that a substantial percentage of the early technical papers on spacecraft dynamics and control involve the application of Lagrange's equations (Refs. 1–6); a few such papers are based on Hamilton's canonical equations (Refs. 7, 8); and in more recent papers several authors pay homage to Hamilton's principle (Refs. 9–11). But the early developments that have provided the basis for most modern simulation practice rely almost invariably upon Newton-Euler formulations (Refs. 12–19).

In recent years the nonrigid spacecraft analysis field has expanded to include many more contributors, and with the diversity of participants has come a variety of approaches to the problem. In this period, progress with literal attitude stability analyses for flexible spacecraft has relied primarily on the concepts of analytical mechanics (see Refs. 20–22, all of which use some form of Hamiltonian as a Liapunov testing function). Lagrange's equations have provided the basis for the rather general nonrigid spacecraft simulation programs developed in Refs. 23–25, although in these papers the analysts have provided a procedure for repeated computer-assembly of equations of motion rather than attempt an explicit generic statement of the necessary equations for the system. Despite these exceptions, it is still true that the dominant formulations are based on the direct application of the methods of Newton and Euler (see Refs. 26–32).

It has, however, been suggested in several recent papers that advantage might be gained by using various Lagrangian or Hamiltonian formulations of the equations of motion, even for large-scale simulations of nonrigid spacecraft (see Refs. 33–35, for examples). The quasi-coordinate approach has been advocated for this role (Ref. 34), and this approach has even been the subject of at least one governmentally funded study contract. Other writers have recently advocated variants of Hamilton's canonical equations for spacecraft simulations, on the basis of the attractiveness of their structure for numerical integration (Ref. 35). The modern quasi-coordinate approach advanced by Kane and Wang (Ref. 36) seems to combine many of the strengths of the Newtonian and Lagrangian approaches, and this method has been proposed for nonrigid spacecraft simulations (Ref. 19), but not systematically evaluated. The utility of Lagrange's equations with Lagrange multipliers is only rarely explored in the literature on space applications (Ref. 37), and one might wonder if the full potential of this approach has been realized.

These many questions are being raised at a time when considerations of economy have mandated careful review of all major efforts within the space program, and dictated a new priority on the development of general-purpose or modular computer programs with the capability of comprehensive but efficient simulation of wide classes of nonrigid spacecraft.

## B. Scope of Study

A general evaluation of the comparative advantages of the procedures of vectorial mechanics (Newton-Euler formulations) and analytical mechanics (Lagrange-Hamilton formulations) would require a monumental effort, and would be almost certain to founder on the rocks of author subjectivity. In this report we consider a much smaller and more specific objective. It is the purpose of this study to explore three of the basic methods of analytical dynamics and their variations, and to examine them for their suitability for the development of multipurpose generic formulations of the equations of motion of nonrigid spacecraft, idealized either as multiple-rigid-body systems or as systems of interconnected elastic or rigid-elastic bodies. Although differences in formulation effort are important also, the final measure of a formulation procedure in the modern context is the accuracy and efficiency of operation of a computer program based on the equations. The present report is intended to assess the advisability of making the commitment of resources necessary to develop the computer programs required if we are to obtain this definitive measure of simulation procedures. The results should be valuable to any readers who face the future prospect of developing computer programs for the simulation of nonrigid spacecraft.

The three methods of analytical dynamics to be developed and evaluated here are as follows:

(1) D'Alembert's principle and its generalizations (including Lagrange's two basic forms of D'Alembert's principle and Kane's more general formulation).

(2) Lagrange's equations (including generalized coordinate equations for both holonomic and simple nonholonomic systems, and quasi-coordinate equations).

(3) Hamilton's equations, in forms applicable to simple nonholonomic and nonconservative systems, as well as the classical restricted cases.

These methods are developed sequentially in Section II of this report, and their applicability to various models of nonrigid spacecraft is examined and compared in Section III. Conclusions and recommendations are contained in the final Section IV.

# II. Selected Methods of Analytical Dynamics

## A. D'Alembert's Principle and Its Generalizations

**1. D'Alembert's principle.** Given a set of $N$ particles,[2] we can use Newton's second law to record a complete set of equations of motion in the form

$$\mathbf{F}_j - m_j \ddot{\mathbf{R}}_j = 0, \qquad j = 1, \cdots, N \tag{1}$$

where $m_j$ is the mass of the $j$th particle, $\mathbf{R}_j$ locates the $j$th particle from an inertially fixed point (and $\ddot{\mathbf{R}}_j$ is the inertial acceleration), and $\mathbf{F}_j$ is the resultant force applied to the particle. As a matter of definition, we can separate $\mathbf{F}_j$ into two parts, $\mathbf{F}_j^e$ and $\mathbf{F}_j^i$, with the latter accommodating the contributions to $\mathbf{F}_j$ of interaction forces within the system of particles. If we then sum the $N$ equations given by Eq. (1), we can observe that the vectors $\mathbf{F}_j^i$ will not appear in the result, due to Newton's third law; thus we have

$$\sum_{j=1}^{N} \mathbf{F}_j^i = 0 \tag{2}$$

and consequently,

$$\sum_{j=1}^{N} (\mathbf{F}_j^e - m_j \ddot{\mathbf{R}}_j) = 0 \tag{3}$$

The term *D'Alembert's principle* is applied in the literature nonuniformly, but Eqs. (1), (2) and (3) are each sometimes given this label. (The original 1743 exposition by D'Alembert in his *Traité de Dynamique* is entitled "A General Principle for Finding the Motions of Several Bodies Which React on Each Other in Any Fashion;" D'Alembert's objective was to establish the disappearance of the interaction forces from Eq. (3) due to Eq. (2). Nonetheless, modern authors have made Eq. (1) familiar to most readers as "D'Alembert's principle.") We cannot digress here for either history or philosophy; we seek only enough information to permit evaluation of the engineering utility of the results. Within this framework, none of the preceding variations of D'Alembert's principle is as useful as an alternative form obtained historically by combining Eq. (1) with Bernoulli's *principle of virtual work*.

Obviously, Eq. (1) can be dot-multiplied by any vector at all and a valid equation will result. In particular, we can introduce the symbols $\delta\mathbf{R}_j$, $j = 1, \cdots, N$ and write Eq. (1) as

$$(\mathbf{F}_j - m_j \ddot{\mathbf{R}}_j) \cdot \delta\mathbf{R}_j = 0, \qquad j = 1, \cdots, N \tag{4}$$

---

[2] We will derive equations for systems consisting of finite numbers of particles, and without formal proof apply our results to continua, replacing particles by differential elements of mass, and summations by integrations. Since we limit scope to systems having constant mass, this transition from discrete to continuous models introduces no difficulties when developed formally.

By summing these $N$ equations, we can obtain

$$\sum_{j=1}^{N} (\mathbf{F}_j - m_j \ddot{\mathbf{R}}_j) \cdot \delta\mathbf{R}_j = 0 \tag{5}$$

Although Eqs. (4) and (5) apply for any definition we may wish to give to $\delta\mathbf{R}_j$ ($j = 1, \cdots, N$), it has been convenient historically to conceive of these vectors as imaginary or *virtual displacements* of the $N$ particles in inertial space; the units or dimensions of the quantities in these equations then correspond to work (e.g., Newton-meter). When the mass times acceleration terms in Eq. (5) are moved to the right hand side of the equation (or considered to be zero, in the static equilibrium case), the resulting equation is said to express the principle of virtual work, and when written as Eq. (5) this equation is sometimes known as the generalized principle of D'Alembert. The practical utility of Eq. (5) is greatly increased when the vectors $\delta\mathbf{R}_j$ are written in terms of *generalized coordinates.*

Any set of $\nu$ scalars $q_1, \cdots, q_\nu$ that fully defines the configuration of the system is called a set of *generalized coordinates.* By this definition, it must be possible to write $\mathbf{R}_j$ in terms of the generalized coordinates and time explicitly, to obtain

$$\mathbf{R}_j = \mathbf{R}_j(q_1, \cdots, q_\nu, t), \qquad j = 1, \cdots, N \tag{6}$$

If now the *virtual displacement* $\delta\mathbf{R}_j$ is interpreted as a *variation* of the vector $\mathbf{R}_j$ in the sense of variational calculus, we can treat the $\delta$ as an operator and employ the expression

$$\delta\mathbf{R}_j = \sum_{k=1}^{\nu} \frac{\partial \mathbf{R}_j}{\partial q_k} \delta q_k, \qquad j = 1, \cdots, N \tag{7}$$

Substituting Eq. (7) into Eq. (5), and reversing the sequence of the finite sums, we find

$$\sum_{k=1}^{\nu} \sum_{j=1}^{N} (\mathbf{F}_j - m_j \ddot{\mathbf{R}}_j) \cdot \frac{\partial \mathbf{R}_j}{\partial q_k} \delta q_k = 0 \tag{8}$$

Eq. (8) is yet another form of D'Alembert's principle; this version was used by Lagrange in the derivation of his famous equations (see Subsection II-B), and it leads to results of direct practical utility in its own right for a wide class of dynamical systems, as we shall see in the following sections.

### 2. Lagrange's form of D'Alembert's principle for independent generalized coordinates

*a. Constraints.* In general the generalized coordinates $q_1, \cdots, q_\nu$ in Eq. (8) are not independent; they may be related by *constraint equations.* If these constraint equations adopt the form

$$h_i(q_1, \cdots, q_\nu, t) = 0, \qquad i = 1, \cdots, m \tag{9}$$

they are said to define *holonomic* constraints, and they can (at least conceptually) be solved for $m$ of the generalized coordinates in terms of the remaining $\nu - m$. (In practice we may wish to forego this option if the constraint equations are not easily solved; then we can use the procedure in the next section.) We designate the number $\nu - m$ as $n$, and speak of $n$ as the number of *degrees of freedom* in the system. (For our system of $N$ particles, we have $n \leq 3N$.)

If we use Eq. (9) to eliminate $m$ of the generalized coordinates in Eq. (6) in favor of the remaining $n$, then Eq. (6) becomes

$$\mathbf{R}_j = \mathbf{R}_j\,(q_1, \cdots, q_n, t) \tag{10}$$

and Eqs. (7) and (8) change correspondingly, with $n$ replacing $\nu$. The important difference lies in the fact that we can allow the generalized virtual displacements $\delta q_1, \cdots, \delta q_n$ to be *independent* without violating constraints, whereas $\delta q_1, \cdots, \delta q_\nu$ must be treated as interdependent unless we want these imaginary displacements to violate the constraints. As will be shown, by maintaining compatibility with constraints we can eliminate certain unknown forces from the equations of motion, so we accept this restriction. When the $n$ generalized virtual displacements are independent, we can select $n$ different sets, each time making all but one equal to zero. Thus we conclude that each of the coefficients of $\delta q_1, \cdots, \delta q_n$ in Eq. (8) is individually zero, to achieve the major objective of this section.

*b. Lagrange's form of D'Alembert's Principle.* For a set of $n$ independent generalized coordinates, Eq. (8) produces the useful result

$$\sum_{j=1}^{N} (\mathbf{F}_j - m_j\,\ddot{\mathbf{R}}_j) \cdot \frac{\partial \mathbf{R}_j}{\partial q_k} = 0, \qquad k = 1, \cdots, n \tag{11}$$

In the form of Eq. (11), Lagrange has written a consequence of D'Alembert's principle and the principle of virtual work that has a substantial practical value, because the system of $n$ equations is in many applications a complete set, sufficient to solve for the kinematical unknowns $q_1, \cdots, q_n$.

*c. Constraint forces.* There may be in addition to the generalized coordinates other unknowns in the physical problem; certain forces may be required to maintain the constraints established by Eq. (9), and these forces will not be known in advance of the solution of the problem. The virtue of Eq. (11) lies in the fact that such unknown *constraint forces*, although present in $\mathbf{F}_j$, very often are absent from Eq. (11). As a matter of definition, we can separate $\mathbf{F}_j$ again into two parts, $\mathbf{f}_j$ and $\mathbf{f}'_j$, with $\mathbf{f}'_j$ representing that part of $\mathbf{F}_j$ that disappears in the course of the dot multiplications and summations in Eq. (11). The forces $\mathbf{f}'_j$ therefore *by definition* satisfy the equation

$$\sum_{j=1}^{N} \mathbf{f}'_j \cdot \frac{\partial \mathbf{R}_j}{\partial q_k} = 0, \qquad k = 1, \cdots, n \tag{12}$$

implying that Eq. (11) takes the form

$$\sum_{j=1}^{N} (\mathbf{f}_j - m_j\,\ddot{\mathbf{R}}_j) \cdot \frac{\partial \mathbf{R}_j}{\partial q_k} = 0, \qquad k = 1, \cdots, n \tag{13}$$

The forces $\mathbf{f}'_j$ are often referred to as *nonworking constraint forces*, because they do no work in the course of a *virtual* displacement compatible with constraints (compare Eq. (12) with Eq. (8), with $\nu$ replaced by $n$). It should *not* be imagined however that the so-called nonworking constraint forces can do no work in the course of the *actual* motion experienced by a physical system (see the example in Ref. 38, pp. 188–190). Nor should we jump to the optimistic conclusion that *all* constraint forces are subsumed by $\mathbf{f}'_j$; a Coulomb friction force, for example, is proportional to the normal force between two bodies, so the magnitude of the unknown normal force of constraint remains in Eq. (11). Nonetheless there is a wide range of commonly considered constraint forces that do classify as nonworking constraint forces, and as long as the problem includes constraint forces of only this kind Eq. (11) provides a complete set of equations that can be solved for $q_1, \cdots, q_n$. In particular, Whittaker[3] shows (Ref. 39, pp. 31–32 and pp. 36–37) that the following classes of forces qualify as nonworking constraint forces when applied to holonomically constrained systems:

(1) The reactions of perfectly smooth or perfectly rough surfaces with which the bodies of the system are constrained to remain in contact, whether these surfaces be inertially fixed or moving in a prescribed manner. The term *smooth* implies that the reaction is normal to the surface, and a *rough* surface is one that precludes sliding and permits only rolling contact.

(2) The mutual reactions of two particles constrained to remain a fixed distance or a prescribed time-varying distance apart. Since a rigid body can be considered as an aggregate of particles interconnected so as to maintain invariable distances between all particle pairs, the internal forces in a rigid body are nonworking constraint forces.

(3) The reactions at any frictionless pinned joint of the system, whether this joint interconnects two bodies of the system or connects one body of the system to an external point which is either inertially stationary or moving in a prescribed manner.

*d. Generalized forces.* It is customary to define the *generalized forces* $Q_1, \cdots, Q_n$ by

$$Q_k \triangleq \sum_{j=1}^{N} \mathbf{F}_j \cdot \frac{\partial \mathbf{R}_j}{\partial q_k} = \sum_{j=1}^{N} \mathbf{f}_j \cdot \frac{\partial \mathbf{R}_j}{\partial q_k}, \qquad k = 1, \cdots, n \tag{14a}$$

where advantage has been taken of Eq. (12) to eliminate $\mathbf{f}'_j$ from $\mathbf{F}_j \triangleq \mathbf{f}_j + \mathbf{f}'_j$. The scalars $Q_k$ are sometimes called *generalized active forces* (Ref. 40), in distinction to the terms

$$Q^*_k \triangleq -\sum_{j=1}^{N} m_j \ddot{\mathbf{R}}_j \cdot \frac{\partial \mathbf{R}_j}{\partial q_k}, \qquad k = 1, \cdots, n \tag{14b}$$

which are then called the *generalized inertia forces.* Eqs. (11) and (13) then take the scalar form

$$Q_k + Q^*_k = 0, \qquad k = 1, \cdots, n \tag{15}$$

[3]Whittaker states that these forces "do no work on the system . . . during the motion" (p. 31), but it is clear from his examples (p. 37) that he means "do no work on the system in the course of a virtual displacement compatible with constraints." Whittaker's list of typical nonworking constraint forces is not identical to that shown here, but it is equivalent.

or the matrix form

$$Q + Q^* = 0 \tag{16}$$

where

$$Q \triangleq \{Q_1, Q_2, \cdots, Q_n\}^T$$
$$Q^* \triangleq \{Q_1^*, Q_2^*, \cdots, Q_n^*\}^T.$$

Eqs. (11), (13), (15), and (16) are all equivalent, and any or all of these equations will be referred to as *Lagrange's form of D'Alembert's principle for independent generalized coordinates.* Since the nonworking constraint forces do not appear in these equations, Eqs. (11), (13), (15), or (16) can provide a complete set of equations of motion for any holonomic system *unless* there exist some unknown constraint forces (such as those normal forces associated with Coulomb friction forces) that do not classify as nonworking constraint forces by the definition in Eq. (12).

In application, the calculation of the generalized forces may be simplified by the identity

$$\frac{\partial \dot{\mathbf{R}}}{\partial \dot{q}_k} = \frac{\partial \mathbf{R}}{\partial q_k} \tag{17}$$

which follows from differentiation of the expansion

$$\dot{\mathbf{R}} \triangleq \frac{d\mathbf{R}}{dt} = \sum_{k=1}^{n} \frac{\partial \mathbf{R}}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{R}}{\partial t} \tag{18}$$

Since $\dot{\mathbf{R}}$ is linear in the generalized velocities, we can obtain $\partial \dot{\mathbf{R}} / \partial \dot{q}_k$ by inspection of $\dot{\mathbf{R}}$, as simply the coefficient of $\dot{q}_k$. Thus for an $N$-particle system we can replace Eqs. (14a) and (14b) by

$$Q_k = \sum_{j=1}^{N} \mathbf{F}_j \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k} = \sum_{j=1}^{N} \mathbf{f}_j \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k}, \qquad k = 1, \cdots, n \tag{19a}$$

$$Q_k^* = -\sum_{j=1}^{N} m_j \ddot{\mathbf{R}}_j \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k}, \qquad k = 1, \cdots, n \tag{19b}$$

and rewrite Eq. (13) in the form

$$\sum_{j=1}^{N} (\mathbf{f}_j - m_j \ddot{\mathbf{R}}_j) \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k} = 0, \qquad k = 1, \cdots, n \tag{20}$$

*e. Rigid bodies.* When a set of particles is so constrained that each particle remains a fixed distance from every other particle, we call it a *rigid body* (notwithstanding the fact that an undeformable material continuum is given the same name). Since there may be many particles in a rigid body, the explicit calculations in Eqs. (19) can become cumbersome. The generalized forces are much more easily calculated if we introduce $\mathbf{R}^p$ as the vector from an inertially fixed point to some (arbitrarily selected) reference point $p$ fixed in the rigid body, and let $\mathbf{r}_j$ locate the $j$th particle with respect to $p$ (see Fig. 1). Then we can substitute

$$\frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k} = \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} + \frac{\partial \dot{\mathbf{r}}_j}{\partial \dot{q}_k} = \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} + \frac{\partial}{\partial \dot{q}_k} (\boldsymbol{\omega} \times \mathbf{r}_j)$$

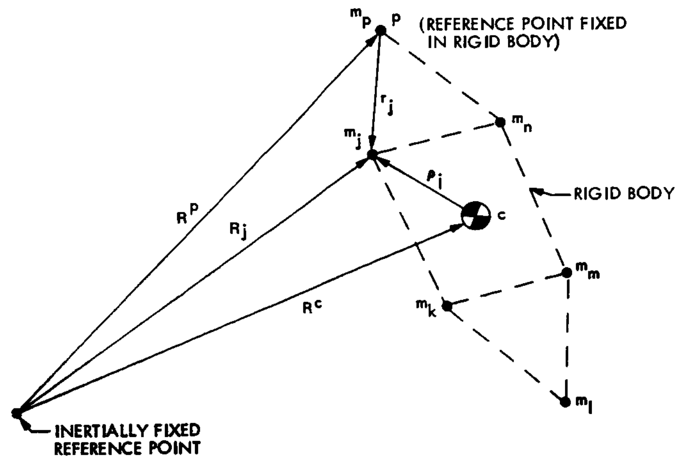


**Fig. 1. Particle system constrained as rigid body**

where $\boldsymbol{\omega}$ is the inertial angular velocity of the rigid body. Since $\mathbf{r}_j$ has no explicit dependence on $\dot{q}_k$, the preceding expression becomes

$$\frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k} = \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} + \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \times \mathbf{r}_j = \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} - \mathbf{r}_j \times \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \tag{21}$$

Thus if all $N$ particles of a system belong to the same rigid body, we can combine this result with Eq. (19) to obtain (after elementary vector operations)

$$Q_k = \sum_{j=1}^{N} \mathbf{f}_j \cdot \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} + \sum_{j=1}^{N} \mathbf{r}_j \times \mathbf{f}_j \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} = \mathbf{f} \cdot \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} + \mathbf{m}^p \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \tag{22}$$

where $\mathbf{f}$ and $\mathbf{m}^p$ are respectively the resultant force and moment for point $p$ applied to the body, excluding nonworking constraint forces defined by Eq. (12). More precisely, these symbols are defined by

$$\begin{aligned} \mathbf{f} &\triangleq \sum_{j=1}^{N} \mathbf{f}_j \\ \mathbf{m}^p &\triangleq \sum_{j=1}^{N} \mathbf{r}_j \times \mathbf{f}_j \end{aligned} \tag{23}$$

Explicit expressions for the generalized inertial force $Q_k^*$ for a rigid body are deferred to the section immediately following.

*f. Systems of particles and rigid bodies.* If the system consists of $\mathcal{P}$ particles and $\mathcal{B}$ rigid bodies, as in Fig. 2, where $\mathcal{B} = 2$ and $\mathcal{P} = 7$, Eq. (19) becomes

$$Q_k = \sum_{j=1}^{\mathcal{P}} \mathbf{f}_j \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k} + \sum_{j=1}^{\mathcal{B}} \left( \mathbf{f}^j \cdot \frac{\partial \dot{\mathbf{R}}^{p_j}}{\partial \dot{q}_k} + \mathbf{m}^{p_j} \cdot \frac{\partial \boldsymbol{\omega}^j}{\partial \dot{q}_k} \right), \qquad k = 1, \cdots, n \tag{24}$$

where $\mathbf{f}^j$ is the resultant force (excluding nonworking constraint forces) on the $j$th body; $\mathbf{R}^{p_j}$ is an inertial position vector locating point $p_j$ on body $j$; and $\mathbf{m}^{p_j}$ is the resultant of moments about $p_j$ of working forces applied to the $j$th body.

In the special case when the reference point $p_j$ is chosen as the mass center $c_j$ of the $j$th body, we will write Eq. (24) in the form

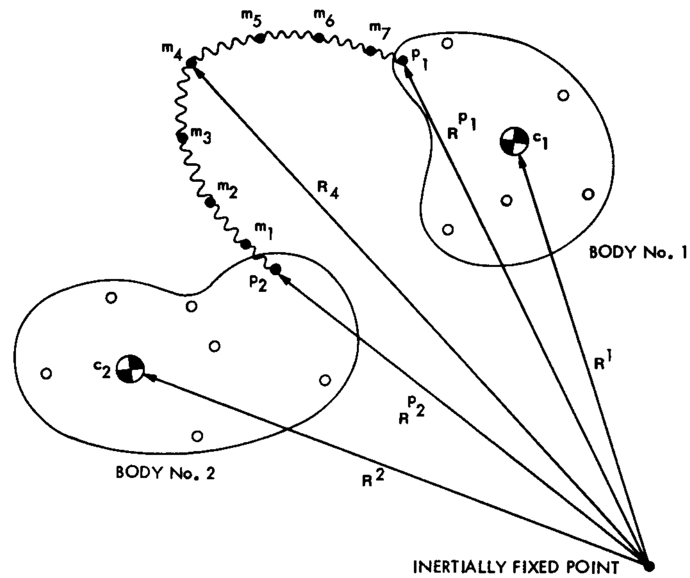


**Fig. 2. System of seven particles and two multiple-particle rigid bodies**

$$Q_k = \sum_{j=1}^{\mathcal{P}} \mathbf{f}_j \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k} + \sum_{j=1}^{\mathcal{B}} \left( \mathbf{f}^j \cdot \frac{\partial \dot{\mathbf{R}}^j}{\partial \dot{q}_k} + \mathbf{m}^j \cdot \frac{\partial \boldsymbol{\omega}^j}{\partial \dot{q}_k} \right), \qquad k = 1, \cdots, n \tag{25}$$

so that $\mathbf{R}^j$ denotes an inertial position vector of the mass center $c_j$ of the $j$th body (see Fig. 2), and $\mathbf{m}^j$ is the moment about the mass center of the $j$th body.

Just as Eq. (19a) led to Eq. (25) as a useful expression for the generalized active force $Q_k$, Eq. (19b) for the generalized inertial force $Q_k^*$ leads for the system of $\mathcal{P}$ particles and $\mathcal{B}$ rigid bodies to

$$Q_k^* = -\sum_{j=1}^{\mathcal{P}} m_j \ddot{\mathbf{R}}_j \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k} - \sum_{j=1}^{\mathcal{B}} \mathcal{M}_j \ddot{\mathbf{R}}^j \cdot \frac{\partial \dot{\mathbf{R}}^j}{\partial \dot{q}_k} - \sum_{j=1}^{\mathcal{B}} \dot{\mathbf{H}}^j \cdot \frac{\partial \boldsymbol{\omega}^j}{\partial \dot{q}_k}, \qquad k = 1, \cdots, n \tag{26}$$

where $\mathcal{M}_j$ is the total mass of the $j$th body and $\mathbf{H}^j$ is the angular momentum of the $j$th body referred to its mass center. (By definition, we have for a rigid body composed of $N_j$ particles

$$\mathbf{H}^j \triangleq \sum_{s=1}^{N_j} m_s \boldsymbol{\rho}_s \times \dot{\boldsymbol{\rho}}_s \tag{27}$$

where as in Fig. 1 $\boldsymbol{\rho}_s$ locates the $s$th particle in the $j$th body with respect to the mass center of that body.) To obtain Eq. (26) from Eq. (19b), we must use Eq. (27) to record

$$\dot{\mathbf{H}}^j = \sum_{s=1}^{N_j} m_s \dot{\boldsymbol{\rho}}_s \times \dot{\boldsymbol{\rho}}_s + \sum_{s=1}^{N_j} m_s \boldsymbol{\rho}_s \times \ddot{\boldsymbol{\rho}}_s = \sum_{s=1}^{N_j} m_s \boldsymbol{\rho}_s \times \ddot{\boldsymbol{\rho}}_s$$
$$= \sum_{s=1}^{N_j} m_s \boldsymbol{\rho}_s \times (\ddot{\mathbf{R}}_s - \ddot{\mathbf{R}}^j) = -\sum_{s=1}^{N_j} m_s \ddot{\mathbf{R}}_s \times \boldsymbol{\rho}_s$$

so we can recognize that

$$\dot{\mathbf{H}}^j \cdot \frac{\partial \boldsymbol{\omega}^j}{\partial \dot{q}_k} = -\sum_{s=1}^{N_j} m_s \ddot{\mathbf{R}}_s \cdot \boldsymbol{\rho}_s \times \frac{\boldsymbol{\omega}^j}{\partial \dot{q}_k}$$

With this substitution, Eq. (26) leads to Eq. (19b). If we combine Eqs. (25) and (26) into Eq. (15) and group the terms judiciously, we obtain

$$\sum_{j=1}^{\mathcal{P}} (\mathbf{f}_j - m_j \ddot{\mathbf{R}}_j) \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k} + \sum_{j=1}^{\mathcal{B}} (\mathbf{f}^j - \mathcal{M}_j \ddot{\mathbf{R}}^j) \cdot \frac{\partial \dot{\mathbf{R}}^j}{\partial \dot{q}_k} + \sum_{j=1}^{\mathcal{B}} (\mathbf{m}^j - \dot{\mathbf{H}}^j) \cdot \frac{\partial \boldsymbol{\omega}^j}{\partial \dot{q}_k} = 0,$$
$$k = 1, \cdots, n \qquad (28)$$

If we had not used Eq. (12) to eliminate nonworking constraint forces from the system, then each of the expressions in parentheses would have been recognizably zero from the Newton-Euler formulation of the mechanics of particles and rigid bodies, since $\mathbf{f}_j$, $\mathbf{f}^j$ and $\mathbf{m}^j$ would then have represented the resultant forces and moments due to all forces applied to the system. In such a case, we need not even be restricted by assumptions of independent generalized coordinates. The value of Eq. (28) over the corresponding (and more general) Newton-Euler formulation lies in the fact that nonworking constraint forces have been eliminated, and the number of equations has been reduced to the smallest number that will provide a complete system of equations for the physical system, assuming that the generalized coordinates are independent and that all unknown forces of constraint qualify as nonworking constraint forces.

*g. Deformable body kinematics.* As noted in footnote 2 at the beginning of Section II, we will accept here without formal argument the transition from particle mechanics to the mechanics of differential elements of mass in a continuum. For such an application, the unknowns become vector functions of space and time rather than a finite number of vectors depending only on time, as in the case of $N$ particles. Thus the equations of motion are formally partial differential equations in spatial and temporal independent variables, rather than the ordinary differential equations in time appearing on the preceding pages.

The analytical description of the motion of a deformable body requires some careful attention to definitions, because the distinctions among the many possible options are rather subtle. Figure 3 provides the first step in the definition of kinematical variables. In this figure the position vector $\mathbf{R}$ locating the typical

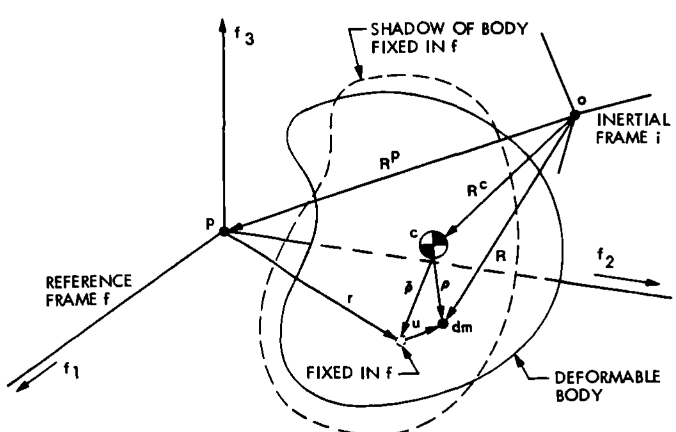


**Fig. 3. Description of a deformable body**

differential element of mass relative to the inertially fixed point o is replaced by the vector sum

$$\mathbf{R} = \mathbf{R}^p + \mathbf{r} + \mathbf{u} \tag{29}$$

The vector $\mathbf{R}^p$ locates relative to o an arbitrary point $p$, which is fixed in an arbitrary reference frame $f$. The differential mass element is then located relative to $p$ by $\mathbf{r} + \mathbf{u}$, where $\mathbf{r}$ is fixed in $f$. Specific choices for $f$ and $p$ are left open, since different options are most attractive in different situations, but always the vector $\mathbf{u}$ describes a displacement of $dm$ from some reference point fixed in $f$. This displacement depends on the material point of the body as well as on time, and we can identify every material point by its position in the reference state, as identified by $\mathbf{r}$. Thus $\mathbf{u}$ is a function of $\mathbf{r}$ and $t$. If we expand $\mathbf{r}$ in terms of unit vectors $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ fixed in frame $f$, so that $\mathbf{r} = r_1\mathbf{f}_1 + r_2\mathbf{f}_2 + r_3\mathbf{f}_3$, we can write

$$\mathbf{u} = \mathbf{u}(r_1, r_2, r_3, t) \tag{30}$$

The kinematical unknowns include not only $\mathbf{u}$, but also six scalar functions of time required for the specification of the motion of $f$, which has not yet been defined uniquely. These six extra unknowns must be specified, either as explicit functions of time, or by six scalar (or two vector) constraint equations, or by a set of six second order ordinary differential equations. We shall entertain various options for equations defining $f$ only when the analytical development provides the motivation for a wise choice.

As noted in conceptual terms previously, the equations of motion include partial differential equations in the vector dependent variable $\mathbf{u}(r_1, r_2, r_3, t)$ in terms of the independent variables $r_1, r_2, r_3, t$. In some important applications, it is possible to formally separate the partial differential equations of vibration of an elastic continuum into an infinite set of ordinary differential equations. In such cases, it is commonplace (and completely reasonable in applied work) to ignore most of the resulting ordinary differential equations, retaining only those with probable significance to the problem under consideration.

The procedure of separating partial differential equations into infinite sets of ordinary differential equations and then truncating them to a finite set is equivalent to the imposition of constraints on the continuum, which restricts its deformation to a finite number of possible modes, with the magnitude of the deformation in each mode being represented by a generalized coordinate. Analytically, this means that the unknown vector function $\mathbf{u} = \mathbf{u}(r_1, r_2, r_3, t)$ is being replaced by the expansion

$$\mathbf{u}(r_1, r_2, r_3, t) = \sum_{j=1}^{\bar{n}} \boldsymbol{\phi}^j (r_1, r_2, r_3)\, q_j(t) \tag{31}$$

where the vector functions of spatial variables $\boldsymbol{\phi}^j$ are specified and the scalars $q_j(t)$ remain as kinematical unknowns, limited in number to $\bar{n}$.

In recognition of this equivalence, an analyst can reasonably decide to idealize his continuum in the first place as a system having a finite number of degrees of freedom in deformation (say $\bar{n}$), represented by the generalized coordinates $q_1, \cdots, q_{\bar{n}}$. The vectors ("mode shapes") $\boldsymbol{\phi}^1, \cdots, \boldsymbol{\phi}^{\bar{n}}$ then become an inherent part of the idealization, and the analyst can formulate ordinary differential equations of motion in the unknown generalized coordinates without giving further thought to the *modal vectors* $\boldsymbol{\phi}^j$ $(j = 1, \cdots, \bar{n})$.

When the time comes to apply the equations of motion to a physical system, the modal vectors must of course be specified. We have noted that for special cases (such as small-deformation beams) one can obtain these functions from the partial differential equations of motion. In some cases it might be acceptable to simply *assign* modal vectors based on engineering judgment. In general, however, both of these options are closed, and one must reject the continuum model in favor of a discretized model, at least for purposes of modal analysis (Ref. 41). The problems of modal analysis of discretized models are treated extensively in Refs. 19 and 27, and they will not be examined here. For present purposes, we shall assume that somehow the mode shapes are known.

*h. Deformable body dynamics.* Having established the general feasibility of describing approximately the motions of a continuum with a finite number of generalized coordinates, we can quickly move to adopt for its equations of motion the results developed as Eq. (15), namely

$$Q_k + Q_k^* = 0, \qquad k = 1, \cdots, n \tag{15}$$

provided that the $n$ generalized coordinates of the system $q_1, \cdots, q_n$ (including now the $\bar{n}$ deformation coordinates) are independent. We can consider these equations of motion only if we revise the generalized force definitions in Eqs. (14) to accommodate the continuum, replacing summations by integrations and particle masses and forces by differential quantities. These extended definitions of generalized active forces $Q_k$ and generalized inertia forces $Q_k^*$ then become

$$Q_k \triangleq \int \left(\frac{\partial \mathbf{R}}{\partial q_k}\right) \cdot d\mathbf{f}, \qquad k = 1, \cdots, n \tag{32a}$$

and

$$Q_k^* \triangleq - \int \left(\frac{\partial \dot{\mathbf{R}}}{\partial \dot{q}_k}\right) \cdot \ddot{\mathbf{R}}\, dm, \qquad k = 1, \cdots, n \tag{32b}$$

As in the case of a rigid body, we find that with some manipulation we can reduce the definitions of generalized forces to simpler forms. For the deformable body, we can combine Eqs. (32a), (29), and (17) to obtain

$$\begin{aligned} Q_k &= \int \left(\frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} + \frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}_k} + \frac{\partial \dot{\mathbf{u}}}{\partial \dot{q}_k}\right) \cdot d\mathbf{f} \\ &= \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} \cdot \mathbf{f} + \int \frac{\partial}{\partial \dot{q}_k} (\boldsymbol{\omega} \times \mathbf{r}) \cdot d\mathbf{f} + \int \frac{\partial}{\partial \dot{q}_k} (\mathring{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{u}) \cdot d\mathbf{f} \end{aligned}$$

where by definition $\boldsymbol{\omega}$ is the inertial angular velocity of frame $f$,

$$\mathbf{f} \triangleq \int d\mathbf{f} \tag{33}$$

and an open circle over the vector $\mathbf{u}$ implies its time differentiation in frame $f$. Noting that

$$\frac{\partial \mathbf{r}}{\partial \dot{q}_k} = \frac{\partial \mathbf{u}}{\partial \dot{q}_k} = 0$$

exchanging dot and cross products in scalar triple products, and extracting $\boldsymbol{\omega}$ from the integral produce

$$Q_k = \mathbf{f} \cdot \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} + \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \cdot \left[ \int \mathbf{r} \times d\mathbf{f} + \int \mathbf{u} \times d\mathbf{f} \right] + \int \frac{\partial \mathring{\mathbf{u}}}{\partial \dot{q}_k} \cdot d\mathbf{f} \tag{34}$$

The nominal moment resultant for $p$ of all forces other than nonworking constraint forces we define as

$$\mathbf{m}^p \triangleq \int \mathbf{r} \times d\mathbf{f} \tag{35}$$

noting that internal forces almost always disappear from this integral. With this substitution and the introduction into $Q_k$ of the expansion in Eq. (31), we find

$$\begin{aligned} Q_k = \mathbf{f} \cdot \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} + \mathbf{m}^p \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} + \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \cdot \sum_{j=1}^{n} \int \boldsymbol{\varphi}^j \times d\mathbf{f}\, q_j \\ + \int \boldsymbol{\varphi}^k \cdot d\mathbf{f}, \qquad k = 1, \cdots, n \end{aligned}$$

where we have assumed that $\boldsymbol{\varphi}^j$ is given as a constant vector in reference frame $f$. In comparing this expression for $Q_k$ to Eq. (22), you can see that we could have obtained $Q_k$ for a rigid body as a special case of $Q_k$ for a deformable body.

The generalized inertia force can be similarly expanded, to obtain from Eq. (32b)

$$\begin{aligned} Q_k^* &= -\int (\ddot{\mathbf{R}}^p + \ddot{\mathbf{r}} + \ddot{\mathbf{u}}) \cdot \left( \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} + \frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}_k} + \frac{\partial \dot{\mathbf{u}}}{\partial \dot{q}_k} \right) dm \\ &= -\ddot{\mathbf{R}}^p \cdot \left[ \mathcal{M} \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} + \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \times \int \mathbf{r}\, dm + \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \times \int \mathbf{u}\, dm + \int \frac{\partial \mathring{\mathbf{u}}}{\partial \dot{q}_k} dm \right] \\ &\quad - \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} \cdot \left[ \boldsymbol{\omega} \times \left( \boldsymbol{\omega} \times \int \mathbf{r}\, dm \right) + \dot{\boldsymbol{\omega}} \times \int \mathbf{r}\, dm \right] \\ &\quad - \int \left\{ [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}] \cdot \left( \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \times \mathbf{r} \right) \right\} dm \\ &\quad - \int \left\{ [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}] \cdot \left( \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \times \mathbf{u} + \frac{\partial \mathring{\mathbf{u}}}{\partial \dot{q}_k} \right) \right\} dm \\ &\quad - \int \left\{ [\mathring{\mathring{\mathbf{u}}} + 2\boldsymbol{\omega} \times \mathring{\mathbf{u}} + \dot{\boldsymbol{\omega}} \times \mathbf{u} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{u})] \cdot \left[ \frac{\partial \dot{\mathbf{R}}^p}{\partial \dot{q}_k} + \frac{\partial \mathring{\mathbf{u}}}{\partial \dot{q}_k} \right.\right. \\ &\quad \left.\left. + \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \times (\mathbf{r} + \mathbf{u}) \right] \right\} dm, \quad k = 1, \cdots, n \end{aligned} \tag{36}$$

The complexity of this expression provides abundant motivation to seek special values of $p$ and $f$ that will simplify $Q_k^*$. In particular, if $p$ is the body mass center for all $\mathbf{u}$, we have

$$\int \mathbf{r}\, dm = \int \mathbf{u}\, dm = \int \mathring{\mathbf{u}}\, dm = 0 \tag{37}$$

and $Q_k^*$ becomes much simpler. Rather than attempt to specialize Eq. (36), however, we will return to the original definition of $Q_k^*$ in Eq. (32b) and accomplish the expansion anew, this time replacing $\mathbf{R}$ by $\mathbf{R}^c + \boldsymbol{\rho} = \mathbf{R}^c + \bar{\boldsymbol{\rho}} + \mathbf{u}$ when appropriate (see Fig. 3). Now we have

$$
\begin{aligned}
Q_k^* &\triangleq -\int \frac{\partial \mathbf{R}}{\partial q_k} \cdot \ddot{\mathbf{R}}\, dm = -\int \ddot{\mathbf{R}} \cdot \frac{\partial \dot{\mathbf{R}}}{\partial \dot{q}_k}\, dm \\
&= -\int \ddot{\mathbf{R}} \cdot \left[ \frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_k} + \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \times (\bar{\boldsymbol{\rho}} + \mathbf{u}) + \frac{\partial \mathring{\mathbf{u}}}{\partial \dot{q}_k} \right] dm \\
&= -\ddot{\mathbf{R}}^c \cdot \mathcal{M} \frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_k} - \int (\ddot{\bar{\boldsymbol{\rho}}} + \ddot{\mathbf{u}})\, dm \cdot \frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_k} + \int \ddot{\mathbf{R}} \times (\bar{\boldsymbol{\rho}} + \mathbf{u}) \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k}\, dm \\
&\quad - \int (\ddot{\mathbf{R}}^c + \ddot{\bar{\boldsymbol{\rho}}} + \ddot{\mathbf{u}}) \cdot \frac{\partial \mathring{\mathbf{u}}}{\partial \dot{q}_k}\, dm \\
&= -\mathcal{M} \ddot{\mathbf{R}}^c \cdot \frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_k} + 0 + \ddot{\mathbf{R}}^c \times \int (\bar{\boldsymbol{\rho}} + \mathbf{u})\, dm \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \\
&\quad - \int (\bar{\boldsymbol{\rho}} + \mathbf{u}) \times (\ddot{\bar{\boldsymbol{\rho}}} + \ddot{\mathbf{u}})\, dm \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} - \ddot{\mathbf{R}}^c \cdot \frac{\partial}{\partial \dot{q}_k} \int \mathring{\mathbf{u}}\, dm \\
&\quad - \int (\ddot{\bar{\boldsymbol{\rho}}} + \ddot{\mathbf{u}}) \cdot \frac{\partial \mathring{\mathbf{u}}}{\partial \dot{q}_k}\, dm \\
&= -\mathcal{M} \ddot{\mathbf{R}}^c \cdot \frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_k} - \frac{d}{dt} \int (\bar{\boldsymbol{\rho}} + \mathbf{u}) \times (\dot{\bar{\boldsymbol{\rho}}} + \dot{\mathbf{u}})\, dm \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \\
&\quad - \int (\ddot{\bar{\boldsymbol{\rho}}} + \ddot{\mathbf{u}}) \cdot \frac{\partial \mathring{\mathbf{u}}}{\partial \dot{q}_k}\, dm \\
&= -\mathcal{M} \ddot{\mathbf{R}}^c \cdot \frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_k} - \dot{\mathbf{H}} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} - \int \frac{\partial \mathring{\mathbf{u}}}{\partial \dot{q}_k} \cdot (\ddot{\bar{\boldsymbol{\rho}}} + \ddot{\mathbf{u}})\, dm
\end{aligned} \tag{38}
$$

where by definition the angular momentum referred to the mass center is

$$
\mathbf{H} \triangleq \int (\bar{\boldsymbol{\rho}} + \mathbf{u}) \times (\dot{\bar{\boldsymbol{\rho}}} + \dot{\mathbf{u}})\, dm \tag{39}
$$

In confirmation of these results, one can combine Eqs. (38), (34), and (15) to obtain (substituting $c$ for $p$)

$$
\begin{aligned}
&[\mathbf{f} - \mathcal{M} \ddot{\mathbf{R}}^c] \cdot \frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_k} + \left[ \int (\bar{\boldsymbol{\rho}} + \mathbf{u}) \times d\mathbf{f} - \dot{\mathbf{H}} \right] \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} \\
&+ \left[ \int \frac{\partial \mathring{\mathbf{u}}}{\partial \dot{q}_k} \cdot (d\mathbf{f} - \ddot{\bar{\boldsymbol{\rho}}}\, dm - \ddot{\mathbf{u}}\, dm) \right] = 0, \qquad k = 1, \cdots, n
\end{aligned} \tag{40}
$$

If we recall that $d\mathbf{f}$ and $\mathbf{f}$ can be replaced by $d\mathbf{F}$ and $\mathbf{F}$ without changing the result (by virtue of Eq. (12)), then the three expressions in brackets can be recognized as individually zero simply by virtue of Newton's laws. In fact, we can now see that with these substitutions Eq. (40) is valid even if we don't have independent generalized coordinates; this equation applies for absolutely any meaning that might be given to the symbols $q$ and $\boldsymbol{\omega}$. But only for a holonomic system that has been characterized in terms of independent generalized coordinates can we be sure that by substituting Eq. (31) into Eq. (40) we can obtain a complete set of equations free of nonworking constraint forces. This is the strength of Lagrange's form of D'Alembert's principle. The combination of Eqs. (31) and (40) produces for a deformable body characterized by $n$ generalized coordinates, of which $\bar{n}$ are deformational coordinates, the following result:

$$[\mathbf{f} - \mathcal{M}\ddot{\mathbf{R}}^c] \cdot \frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_k} + \left[\mathbf{m} - \dot{\mathbf{H}} + \sum_{j=1}^{\bar{n}} q_j \int \boldsymbol{\varphi}^j \times d\mathbf{f}\right] \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k} + \int \boldsymbol{\varphi}^k \cdot d\mathbf{f}$$
$$- \int \boldsymbol{\varphi}^k \cdot \left\{\sum_{j=1}^{\bar{n}} \boldsymbol{\varphi}^j \ddot{q}_j + 2\boldsymbol{\omega} \times \sum_{j=1}^{\bar{n}} \boldsymbol{\varphi}^j \dot{q}_j\right.$$
$$\left. + \dot{\boldsymbol{\omega}} \times \left(\bar{\boldsymbol{\rho}} + \sum_{j=1}^{\bar{n}} \boldsymbol{\varphi}^j q_j\right) + \boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times \left(\bar{\boldsymbol{\rho}} + \sum_{j=1}^{\bar{n}} \boldsymbol{\varphi}^j q_j\right)\right]\right\} dm = 0, \qquad k = 1, \cdots, n \tag{41}$$

in which $\mathbf{m} \triangleq \int \bar{\boldsymbol{\rho}} \times d\mathbf{f}$ is the moment about $c$ of forces applied to the undeformed body. (See Eq. (35).) It is important to remember that $\boldsymbol{\varphi}^k$ describes deformation relative to frame $f$; hence $\boldsymbol{\varphi}^k$ is zero if $q_k$ is a discrete coordinate describing only the motion of $f$.

*i. Systems of particles, rigid bodies, and deformable bodies.* By referring to Eqs. (28) and (41), one can assemble the equations of motion of a system of $\mathcal{P}$ particles and $\mathcal{B}$ extended bodies, of which the subset $\bar{\mathcal{B}}$ are deformable, as follows. Here $n$ is again the number of independent generalized coordinates characterizing the total system, and $\bar{n}_j$ is the number of deformational coordinates of the $j$th body.

$$\sum_{j=1}^{\mathcal{P}} (\mathbf{f}_j - m_j \ddot{\mathbf{R}}_j) \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k} + \sum_{j=1}^{\mathcal{B}} (\mathbf{f}^j - \mathcal{M}_j \ddot{\mathbf{R}}^j) \cdot \frac{\partial \dot{\mathbf{R}}^j}{\partial \dot{q}_k} + \sum_{j=1}^{\mathcal{B}} (\mathbf{m}^j - \dot{\mathbf{H}}^j) \cdot \frac{\partial \boldsymbol{\omega}^j}{\partial \dot{q}_k}$$
$$+ \sum_{j=1}^{\bar{\mathcal{B}}} \sum_{\alpha=1}^{\bar{n}_j} q_\alpha \int \boldsymbol{\varphi}^\alpha \times d\mathbf{f} \cdot \frac{\partial \boldsymbol{\omega}^j}{\partial \dot{q}_k} + \sum_{j=1}^{\bar{\mathcal{B}}} \left\{\int_j \boldsymbol{\varphi}^k \cdot d\mathbf{f} - \int_j \boldsymbol{\varphi}^k \cdot \left\{\sum_{\alpha=1}^{\bar{n}_j} \boldsymbol{\varphi}^\alpha \ddot{q}_\alpha + 2\boldsymbol{\omega}^j \times \sum_{\alpha=1}^{\bar{n}_j} \boldsymbol{\varphi}^\alpha \dot{q}_\alpha\right.\right.$$
$$\left.\left. + \dot{\boldsymbol{\omega}}^j \times \left(\bar{\boldsymbol{\rho}} + \sum_{\alpha=1}^{\bar{n}_j} \boldsymbol{\varphi}^\alpha q_\alpha\right) + \boldsymbol{\omega}^j \times \left[\boldsymbol{\omega}^j \times \left(\bar{\boldsymbol{\rho}} + \sum_{\alpha=1}^{\bar{n}_j} \boldsymbol{\varphi}^\alpha q_\alpha\right)\right]\right\} dm\right\}, \qquad k = 1, \cdots, n \tag{42}$$

*j. Floating reference frames.* It is important to remember that we have, at this point, not yet made a commitment to a particular choice of the reference frame $f_j$, with respect to which the deformation of the $j$th body is measured. Our only concession has been to fix the mass center $c_j$ of the $j$th body in $f_j$; this step was taken when, after Eq. (37), we replaced the arbitrary point $p$ with $c$ to simplify our expression for generalized inertial forces. Now we can see that we can exercise our remaining freedom in the selection of the floating reference frames in such a way as to further simplify the equations of motion by imposing as a constraint a single vector equation (or three scalar equations). This constraint is usually chosen in such a way as to simplify the angular momentum of the deformable body referred to its mass center, which is designated as **H** in Eq. (41). Eq. (39) provides for **H** the expression

$$\mathbf{H} \triangleq \int (\bar{\boldsymbol{\rho}} + \mathbf{u}) \times (\dot{\bar{\boldsymbol{\rho}}} + \dot{\mathbf{u}})\, dm = \int (\bar{\boldsymbol{\rho}} + \mathbf{u}) \times [\boldsymbol{\omega} \times (\bar{\boldsymbol{\rho}} + \mathbf{u})]\, dm$$

$$+ \int (\bar{\boldsymbol{\rho}} + \mathbf{u}) \times \mathring{\mathbf{u}}\, dm = \mathbf{I} \cdot \boldsymbol{\omega} + \int (\bar{\boldsymbol{\rho}} + \mathbf{u}) \times \mathring{\mathbf{u}}\, dm \tag{43}$$

where

$$\mathbf{I} \triangleq \int [(\bar{\boldsymbol{\rho}} + \mathbf{u}) \cdot (\bar{\boldsymbol{\rho}} + \mathbf{u})\, \mathbf{U} - (\bar{\boldsymbol{\rho}} + \mathbf{u})(\bar{\boldsymbol{\rho}} + \mathbf{u})]\, dm$$

with **U** the unit dyadic.

One obvious choice of a constraint equation is

$$\int (\bar{\boldsymbol{\rho}} + \mathbf{u}) \times \mathring{\mathbf{u}}\, dm = 0 \tag{44}$$

implying that the reference frame is chosen such that time varying deformations (represented by $\mathring{\mathbf{u}}$) make no contribution to the angular momentum about the system mass center. Thus the relative angular momentum of the body with respect to this frame is zero. We designate this reference frame as $f_T$, and call it the *Tisserand frame*, following the convention in astronomy (Ref. 2). In terms of scalar components of $\bar{\boldsymbol{\rho}}$ and $\mathbf{u}$ in a dextral orthogonal vector basis fixed in $f_T$, Eq. (44) is equivalent to the three equations

$$\frac{d}{dt}\int (\bar{\rho}_2 u_3 - \bar{\rho}_3 u_2)\, dm + \int (u_2 \dot{u}_3 - u_3 \dot{u}_2)\, dm = 0 \tag{45a}$$

$$\frac{d}{dt}\int (\bar{\rho}_3 u_1 - \bar{\rho}_1 u_3)\, dm + \int (u_3 \dot{u}_1 - u_1 \dot{u}_3)\, dm = 0 \tag{45b}$$

$$\frac{d}{dt}\int (\bar{\rho}_1 u_2 - \bar{\rho}_2 u_1)\, dm + \int (u_1 \dot{u}_2 - u_2 \dot{u}_1)\, dm = 0 \tag{45c}$$

With these constraints, the system angular momentum is simply

$$\mathbf{H} = \mathbf{I} \cdot \boldsymbol{\omega} \tag{46}$$

An alternative to the Tisserand frame that provides simpler constraint equations but results in less simplification of the equations of motion is used by Buckens (Ref. 3) in the form

$$\int \bar{\boldsymbol{\rho}} \times \mathbf{u}\, dm = 0 \tag{47}$$

This constraint has the consequence

$$\int \bar{\boldsymbol{\rho}} \times \mathring{\mathbf{u}}\, dm = 0$$

so that the system angular momentum becomes

$$\mathbf{H} = \mathbf{I} \cdot \boldsymbol{\omega} + \int \mathbf{u} \times \mathring{\mathbf{u}}\, dm \tag{48}$$

If second degree terms in $\mathbf{u}$ and $\mathring{\mathbf{u}}$ are neglected in $\mathbf{H}$, or if deformations are restricted in such a way that $\mathbf{u} \times \mathring{\mathbf{u}} = 0$, this frame (which we'll call the Buckens frame $f_B$) can also be interpreted as the frame with respect to which relative angular momentum about the mass center is zero. (This is the interpretation offered by Milne in Ref. 42, for example.) As will become apparent in Subsection B-1, the second degree deformation terms appearing in the integral in Eq. (48) appear also in the kinetic energy expression, where they can contribute linear terms in deformation to the equations of motion. For this reason, it remains important to distinguish between the Buckens frame $f_B$ and the Tisserand frame $f_T$.

In terms of scalar components of $\bar{\boldsymbol{\rho}}$ and $\mathbf{u}$ for a dextral, orthogonal vector basis fixed in the Buckens' frame, Eq. (47) becomes

$$\int (\bar{\rho}_2 u_3 - \bar{\rho}_3 u_2)\, dm = 0 \tag{49a}$$

$$\int (\bar{\rho}_3 u_1 - \bar{\rho}_1 u_3)\, dm = 0 \tag{49b}$$

$$\int (\bar{\rho}_1 u_2 - \bar{\rho}_2 u_1)\, dm = 0 \tag{49c}$$

A third common choice for the floating reference frame is the *principal axis frame* (here designated $f_P$), in which the principal axes of inertia of the deforming body for the mass center remain fixed. If dextral, orthogonal unit vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, fixed in the principal axis frame $f_P$, are assembled in the column array $\{\mathbf{p}\}$, then the inertia dyadic $\mathbf{I}$ appearing in Eq. (43) has the property that the inertia matrix $\bar{I} \triangleq \{\mathbf{p}\} \cdot \mathbf{I} \cdot \{\mathbf{p}\}^T$ is diagonal. In vector-dyadic terms, this means that

$$\mathbf{p}_1 \cdot \mathbf{I} \cdot \mathbf{p}_2 = \mathbf{p}_2 \cdot \mathbf{I} \cdot \mathbf{p}_3 = \mathbf{p}_3 \cdot \mathbf{I} \cdot \mathbf{p}_1 = 0 \tag{50a}$$

These three scalar equations can be written in matrix terms as

$$U^{jT} \bar{I} U^k = U^{jT} \int [(\bar{\rho}^T + u^T)(\bar{\rho} + u)U - (\bar{\rho} + u)(\bar{\rho} + u)^T]\, dm\, U^k = 0, \qquad j \neq k \tag{50b}$$

where

$$U^1 \triangleq [100]^T$$

$$U^2 \triangleq [010]^T$$

$$U^3 = [001]^T$$

and $\bar{\rho}$ and $u$ represent vectors $\bar{\boldsymbol{\rho}}$ and $\mathbf{u}$ in vector basis $\{\mathbf{p}\}$. Since

$$U^{jT} U U^k = U^{jT} U^k = 0$$

for $j \neq k$, Eq. (50b) has the scalar implications

$$\int (\bar{\rho}_j + u_j)(\bar{\rho}_k + u_k)\, dm = 0, \qquad j \neq k \tag{50c}$$

where $\bar{\rho}_j \triangleq \bar{\boldsymbol{\rho}} \cdot \mathbf{p}_j$, and so forth. Eq. (50c) applies for all values of $u_j$ and $u_k$, including zero. Thus Eq. (50c) has the further implication

$$\int (\bar{\rho}_j u_k + u_j \bar{\rho}_k + u_j u_k)\, dm = 0, \qquad j \neq k; j, k = 1, 2, 3 \tag{50d}$$

Because Milne (Ref. 42) ignores second degree terms such as $u_j u_k$, his representation of the principal axis frame constraint equations differs from Eq. (50d) in the absence of these terms. As noted previously, such second degree terms, when appearing in an expression for kinetic energy, can give rise to first degree terms in Lagrange's equations of motion (as developed in Subsection B-1). For this reason it may be necessary to distinguish Milne's approximate principal axis frame from that defined by Eq. (50d).

Thus we see that there exists a variety of reference floating frames that appear in the literature as measures of the "mean motion" or "gross motion" of a deformable body, and with respect to which deformations are measured. Since this choice of reference frame is *implicit* in the work of many authors (who may only speak vaguely of "rigid body modes"), it is important to recognize that there is no universal or uniquely advantageous choice for this frame. For this reason we have left the choice open in recording Eq. (42) as our final version of Lagrange's form of D'Alembert's principle for systems of particles, rigid bodies, and deformable bodies characterized by independent generalized coordinates. We will see in subsequent sections dealing with deformable bodies that the question of establishing a floating reference frame continues to arise, but we will not make a specific choice in this report until specific problems are considered.

*k. An example.* Because confusion often surrounds the floating reference frame and the manner in which the constraint equations defining this frame are used in formulating equations of motion, a brief illustration may be worthwhile. For this purpose we consider the combination of a deformable body and two identical rigid bodies shown in its undeformed state in Fig. 4a. Milne (Ref. 42) uses this example to distinguish between the Tisserand (zero relative momentum) frame and the instantaneous principal axis frame by sketching the deformed shape shown in Fig. 4b and labeling axes fixed in the two reference frames. By inspection of the deformed shape one can readily see that the principal axes must, in a qualitative sense, be as portrayed by dashed lines in Fig. 4b; that is, for this pattern of beam vibration, the two rigid bodies must be in the second and fourth quadrants, on opposite sides of the $\mathbf{p}_1$ axis through the mass center $c$, and cannot lie directly on that axis or in the first and third quadrants. In contrast, the Tisserand frame axes *can* adopt the orientation relative to the deformed body suggested by the solid axes in Fig. 4b, with the rigid bodies falling into the first and third quadrants established by these axes. For a special geometry and mass distribution, the axis $\mathbf{T}_1$ would pass through the mass centers of the rigid bodies, and as the mass center moment of inertia of each of these rigid bodies diminishes relative to that of the beam about its mass center, the solid (Tisserand) axes must approach alignment with the dashed (principal) axes.

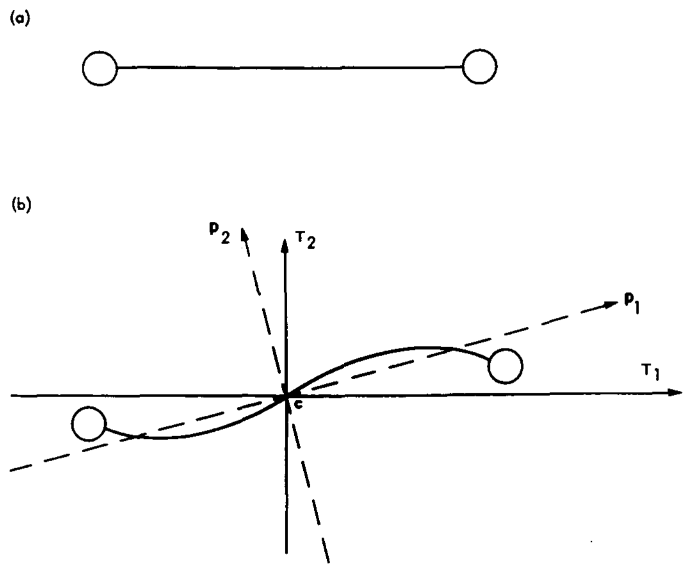


**Fig. 4. An example illustrating two floating reference frames: (a) undeformed; (b) single-mode vibration**

It is physically possible for the system in Fig. 4a to respond to an initial deformation of the sort portrayed in Fig. 4b (with no initial inertial velocities) in such a way that all parts of the system move harmonically in phase at the same frequency relative to the Tisserand frame, while that frame remains inertially stationary. In the course of such a motion, the axes of the principal axis frame must experience an angular oscillation in inertial space at the same frequency, following the motions of the deforming body.

To see exactly how the constraint equations defining the reference frame are used in a dynamical description of a typical mechanical system, we will record the equations of motion of the system in Fig. 4 twice, using first the Tisserand frame and then the principal axis frame. In each case we will consider the force-free, torque-free motion, and permit only the single mode of deformation depicted in Fig. 4b. Thus the system has seven independent degrees of freedom (six for the frame and one for the motion relative to the frame).

Without yet designating either $f_T$ or $f_P$ as the floating reference frame, we can write equations of motion for our system from Eq. (41) as follows (noting that $\mathbf{f} = \mathbf{m} = 0$):

$$-\mathcal{M}\ddot{\mathbf{R}}^c\cdot\frac{\partial\dot{\mathbf{R}}^c}{\partial\dot{q}_k} - \dot{\mathbf{H}}\cdot\frac{\partial\boldsymbol{\omega}}{\partial\dot{q}_k} - \int\boldsymbol{\phi}^k\cdot\{\boldsymbol{\phi}^1\ddot{q}_1 + 2\boldsymbol{\omega}\times\boldsymbol{\phi}^1\dot{q}_1 + \dot{\boldsymbol{\omega}}\times(\bar{\boldsymbol{\rho}} + \boldsymbol{\phi}^1 q_1)$$
$$+ \boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\bar{\boldsymbol{\rho}} + \boldsymbol{\phi}^1 q_1)]\}\,dm = 0, \qquad k = 1,\cdots,7 \tag{51}$$

where $q_1$ is the single distributed deformation coordinate, and $\boldsymbol{\phi}^1$ the corresponding mode shape. If we further identify $q_2, q_3, q_4$ as the 1-2-3 attitude angles of the frame $f$, and $q_5, q_6, q_7$ as the cartesian coordinates of mass center $c$ in inertial space, so that in terms of inertially fixed unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ we have $\dot{\mathbf{R}}^c = \dot{q}_5\mathbf{i}_1 + \dot{q}_6\mathbf{i}_2 + \dot{q}_7\mathbf{i}_3$, then the equations of mass center translation take the trivial form

$$\ddot{\mathbf{R}}^c \cdot \mathbf{i}_1 = \ddot{q}_5 = 0 \qquad (52a)$$

$$\ddot{\mathbf{R}}^c \cdot \mathbf{i}_2 = \ddot{q}_6 = 0 \qquad (52b)$$

$$\ddot{\mathbf{R}}^c \cdot \mathbf{i}_3 = \ddot{q}_7 = 0 \qquad (52c)$$

regardless of the choice of $f$. The system rotational equations become, for either choice of $f$,

$$\dot{\mathbf{H}} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_2} = 0 \qquad (52d)$$

$$\dot{\mathbf{H}} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_3} = 0 \qquad (52e)$$

$$\dot{\mathbf{H}} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_4} = 0 \qquad (52f)$$

but this time the symbol $\boldsymbol{\omega}$ has different meanings for different choices of $f$. Recall that $\boldsymbol{\omega}$ is the inertial angular velocity of frame $f$, so it clearly differs for the Tisserand frame and the principal axis frame (see Fig. 4b). Moreover, for the Tisserand frame, $\mathbf{H}$ is merely $\mathbf{I} \cdot \boldsymbol{\omega}$, as in Eq. (46), while for the principal axis frame one must accept Eq. (43) for $\mathbf{H}$. For our example Eq. (43) combines with Eq. (31) to provide

$$\mathbf{H} = \mathbf{I} \cdot \boldsymbol{\omega} + \int (\bar{\boldsymbol{\rho}} + \boldsymbol{\phi}^1 q_1) \times \boldsymbol{\phi}^1 \, dm \, \dot{q}_1 = \mathbf{I} \cdot \boldsymbol{\omega} + \int \bar{\boldsymbol{\rho}} \times \boldsymbol{\phi}^1 \, dm \, \dot{q}_1$$

Only by expanding the inertia dyadic $\mathbf{I}$ in our two expressions for $\mathbf{H}$ can we make visible any advantages in the choice of the principal axis frame; with this selection we know that the inertia matrix is diagonal in the vector basis fixed in $f_P$, while it is generally full for the vector basis fixed in $f_T$.

The seventh and final equation of motion requires the selection $k = 1$. Since $\ddot{\mathbf{R}}^c/\partial \dot{q}_1 = \partial \boldsymbol{\omega}/\partial \dot{q}_1 = 0$ in either case, the equation of vibration becomes

$$0 = \int \boldsymbol{\phi}^1 \cdot \{\boldsymbol{\phi}^1 \ddot{q}_1 + 2\boldsymbol{\omega} \times \boldsymbol{\phi}^1 \dot{q}_1 + \dot{\boldsymbol{\omega}} \times (\bar{\boldsymbol{\rho}} + \boldsymbol{\phi}^1 q_1) + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\bar{\boldsymbol{\rho}} + \boldsymbol{\phi}^1 q_1)]\} \, dm$$

or

$$0 = \int \{\boldsymbol{\phi}^1 \cdot \boldsymbol{\phi}^1 \ddot{q}_1 + \dot{\boldsymbol{\omega}} \cdot (\bar{\boldsymbol{\rho}} \times \boldsymbol{\phi}^1) + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\bar{\boldsymbol{\rho}} + \boldsymbol{\phi}^1 q_1)] \cdot \boldsymbol{\phi}^1\} \, dm$$

or

$$0 = \int \boldsymbol{\phi}^1 \cdot \boldsymbol{\phi}^1 \, dm \, \ddot{q}_1 + \dot{\boldsymbol{\omega}} \cdot \int \bar{\boldsymbol{\rho}} \times \boldsymbol{\phi}^1 \, dm + \boldsymbol{\omega} \cdot \int [\boldsymbol{\omega} \times (\bar{\boldsymbol{\rho}} + \boldsymbol{\phi}^1 q_1)] \times \boldsymbol{\phi}^1 \, dm$$

or

$$0 = \int \boldsymbol{\phi}^1 \cdot \boldsymbol{\phi}^1 \, dm \, \ddot{q}_1 + \dot{\boldsymbol{\omega}} \cdot \int \bar{\boldsymbol{\rho}} \times \boldsymbol{\phi}^1 \, dm - \boldsymbol{\omega} \cdot \int [\boldsymbol{\phi}^1 \cdot \boldsymbol{\phi}^1 \mathbf{U} - \boldsymbol{\phi}^1 \boldsymbol{\phi}^1] \, dm \cdot \boldsymbol{\omega} q_1$$
$$- \boldsymbol{\omega} \cdot \int [\bar{\boldsymbol{\rho}} \cdot \boldsymbol{\phi}^1 \mathbf{U} - \bar{\boldsymbol{\rho}} \boldsymbol{\phi}^1] \, dm \cdot \boldsymbol{\omega} \qquad (52g)$$

This is the equation of vibration for either selection of frame $f$, but the symbols have different meanings in the two cases considered. As shown in Fig. 4b, the "mode shape" $\boldsymbol{\phi}^1$ describing displacement relative to $f_P$ is different than that for $f_T$; consequently the unknowns $\boldsymbol{\omega}$ and $q_1$ will emerge from the system equations as different pairs of functions of time in the two cases. We can shed further light on these differences by examining just how the mode shape $\boldsymbol{\phi}^1$ for $f_P$ differs from that for $f_T$.

For the Tisserand frame $f_T$, the constraint Eq. (44) implies in this case that $\int \bar{\boldsymbol{\rho}} \times \boldsymbol{\phi}^1\, dm = 0$, thereby permitting one integral to be dropped from the vibration Eq. (52g). This integral is not zero if we select the principal axis frame, but instead the inertia dyadic appearing in $\mathbf{H}$ and hence in Eqs. (52d)–(52f) is a diagonal matrix in vector basis $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$.

Thus far in this example we have treated the mode shapes $\boldsymbol{\phi}^1$ as somehow *given* for both the Tisserand and principal axis frames (as depicted in Fig. 4b). But if we plan to solve any real problems we must have a procedure for finding these vectors. In practice they are sometimes merely prescribed on the basis of engineering judgment, without guaranteeing the satisfaction of any of the constraint equations presented here. Then we have no physical meaning attached to the floating reference frame $f$; it is a reference frame whose motion may be fully described by a properly formulated set of equations of motion, and the results may be quite meaningful, but we must remain vague about the definition of the floating frame and allow the equations of motion to define it. (Of course we cannot then simplify $\mathbf{H}$ as for the Tisserand frame or the principal axis frame.)

As noted previously, the mode shapes $\boldsymbol{\phi}^j$ sometimes emerge as eigenvectors of ordinary differential equation sets obtained by separating the partial differential equations of vibration of a continuum. In most engineering practice, the vectors $\boldsymbol{\phi}^j$ are obtained as the eigenvectors of a set of linear constant-coefficient ordinary differential equations that describe the small vibrations of a discretized model of the system. In either case, when $\boldsymbol{\phi}^j$ is obtained formally as an eigenvector independent of all other eigenvectors of the system this has the implication that the system is capable of a motion in which $q_j$ is some function of time and all $q_k$ for $k \neq j$ are zero; thus the equations of vibration in the various "normal modes" of the system are uncoupled.

Among the eigenvectors describing the possible independent motions of an unsupported force-free system there will be six describing translations and rotations of the system without deformations; these are the so-called "rigid body modes." Although the equations of motion for the generalized coordinates corresponding to the rigid body modes are formally excluded from our system equations and replaced by coordinates describing the motions of the frame $f$, it is clear enough that for small motions the rigid body model coordinates describe the motion of some floating frame $f$. For such small motions, one could linearize the vibration equation preceding (dropping second degree terms in the set $\ddot{q}_1, \dot{q}_1, q_1, \dot{\boldsymbol{\omega}}, \boldsymbol{\omega}$) to find

$$\int \boldsymbol{\phi}^1 \cdot \boldsymbol{\phi}^1\, dm\, \ddot{q}_1 + \dot{\boldsymbol{\omega}} \cdot \int \bar{\boldsymbol{\rho}} \times \boldsymbol{\phi}^1\, dm = 0$$

Since the equations of motion in the rigid body modal coordinates (represented by $\boldsymbol{\omega}$ of frame $f$) must be uncoupled from those involving $q_1$, the set of eigenvectors must be such that in this case $\int \bar{\boldsymbol{\rho}} \times \boldsymbol{\phi}^1\, dm = 0$. By referring to Eq. (41) you can confirm that this must be a general property of the eigenvectors that correspond to free vibration and uncouple from motion in the rigid body modes. Thus we conclude that at least for small vibrations the reference frame implied by the rigid body modes from a free vibration modal analysis is the Tisserand frame $f_T$. This is the most compelling advantage for the use of this reference frame.

**3. Lagrange's form of D'Alembert's principle for simply constrained systems.** Equations (8), (11), (13), (16), (20), (28), and (41) of the previous section all represent alternative versions of Lagrange's form of D'Alembert's principle. For present purposes, we can concentrate on Eq. (20), which is a convenient formulation for a system of $N$ particles. Because the $n$ generalized virtual displacements in Eq. (8) are independent, Eq. (20) produces $n$ independent differential equations, which provide a complete set of equations of motion (assuming that all constraints are in the nonworking class). But if the various generalized coordinates were related by a constraint equation, we would not be able to obtain from Eq. (8) as many independent equations as we have generalized coordinates unless we violated constraints with the virtual displacements, and if we violated constraints we would not succeed in our objective of eliminating unknown nonworking constraint forces.

In the present section we will discover that we can circumvent the problem of obtaining a complete set of equations for a wider class of problems than is covered by the material in the preceding section. In this development we will initially focus attention on systems of particles, with the understanding that extension to rigid bodies and continua is a straightforward task that would require repetition of some of the lengthy development of the previous section.

If again we let $q_1, \cdots, q_\nu$ comprise an arbitrary set of generalized coordinates (which need not be independent), then we have $\mathbf{R}_j = \mathbf{R}_j(q_1, \cdots, q_\nu, t)$ as in Eq. (6), and

$$\dot{\mathbf{R}}_j = \sum_{k=1}^{\nu} \frac{\partial \mathbf{R}_j}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{R}_j}{\partial t} \tag{53}$$

The vectors $\partial \dot{\mathbf{R}}_j / \partial \dot{q}_k = \partial \mathbf{R}_j / \partial q_k$ are not independent if the generalized coordinates are related by constraint equations. Suppose however that the constraint equations can be written in the scalar form

$$\sum_{k=1}^{\nu} A_{sk} \dot{q}_k + B_s = 0, \qquad s = 1, \cdots, m \tag{54}$$

or equivalently the matrix form

$$A\dot{q} + B = 0 \tag{55}$$

where $A \triangleq [A_{sk}]$ is an $m$ by $\nu$ matrix, $B \triangleq \{B_s\}$ is an $m$ by 1 matrix, and $\dot{q} \triangleq \{\dot{q}_k\}$ is a $\nu$ by 1 matrix. Holonomic constraint equations (see Eq. (9)) can always be placed in the form of Eqs. (54) and (55) by differentiation, and nonholonomic constraints in the class called *Pfaffian* or *simple* have this structure also. The matrices $A$ and $B$ generally depend on $q$ and $t$ (but not on $\dot{q}$), so that it may be impossible to integrate these constraint equations to find $m$ of the generalized coordinates in terms of the remaining $n$ (where $n \triangleq \nu - m$, as previously). This is always theoretically possible for holonomic constraints, but it may be very difficult even in this special case. It is however always a relatively straightforward task to solve for $m$ of the generalized velocities in terms of the remaining $n$ generalized velocities and the full set of $\nu$ generalized coordinates. We can for example partition Eq. (55) to obtain

$$[A^{mn} \,|\, A^{mm}] \left\{ \frac{\dot{\bar{q}}}{\dot{q}^c} \right\} + \{B\} = \{0\} \tag{56}$$

where $\bar{q}$ has dimension $n$ by 1 and $q^c$ dimension $m$ by 1, with the superscripts on the $A$ partitions signaling their dimensions. Equation (56) is then equivalent to

$$[A^{mm}]\{\dot{q}^c\} = -[A^{mn}]\{\dot{\bar{q}}\} - \{B\}$$

or

$$\{\dot{q}^c\} = -[A^{mm}]^{-1}[A^{mn}]\{\dot{\bar{q}}\} - [A^{mm}]^{-1}\{B\} \tag{57}$$

If now Eq. (53) is rewritten in the form (see Appendix A)

$$\dot{\mathbf{R}}_j = \frac{\partial \mathbf{R}_j}{\partial q^T}\dot{q} + \frac{\partial \mathbf{R}_j}{\partial t} = \frac{\partial \mathbf{R}_j}{\partial \bar{q}^T}\dot{\bar{q}} + \frac{\partial \mathbf{R}_j}{\partial q^{cT}}\dot{q}^c + \frac{\partial \mathbf{R}_j}{\partial t}$$

then we can use Eq. (57) to write $\dot{\mathbf{R}}_j$ in the form

$$\begin{aligned}\dot{\mathbf{R}}_j &= \frac{\partial \mathbf{R}_j}{\partial \bar{q}^T}\dot{\bar{q}} - \frac{\partial \mathbf{R}_j}{\partial q^{cT}}(A^{mm^{-1}}A^{mn}\dot{\bar{q}} + A^{mm^{-1}}B) + \frac{\partial \mathbf{R}_j}{\partial t} \\ &= \left(\frac{\partial \mathbf{R}_j}{\partial \bar{q}^T} - \frac{\partial \mathbf{R}_j}{\partial q^{cT}}A^{mm^{-1}}A^{mn}\right)\dot{\bar{q}} + \left(-\frac{\partial \mathbf{R}_j}{\partial q^{cT}}A^{mm^{-1}}B + \frac{\partial \mathbf{R}_j}{\partial t}\right)\end{aligned} \tag{58}$$

Here we have written $\dot{\mathbf{R}}_j$ as a linear combination of $n$ of the $\nu$ generalized velocities. If we unburden ourselves of the explicit matrix notation in Eq. (58), we can rewrite this expression symbolically in the form

$$\dot{\mathbf{R}}_j = \sum_{k=1}^{n}\mathbf{V}_k^j\dot{q}_k + \mathbf{V}_t^j, \qquad j = 1, \cdots, N \tag{59a}$$

where $\mathbf{V}_k^j$ and $\mathbf{V}_t$ are vectors that may depend on $q_1, \cdots, q_\nu$ and $t$. In practice it is often easiest to obtain these vectors by using ad hoc procedures to find $\dot{\mathbf{R}}_j$ in terms of only $n$ of the original $\nu$ generalized velocities, and then identifying the coefficient of $\dot{q}_k$ as $\mathbf{V}_k^j$.

If we compare Eq. (59a, b) to Eq. (18), it becomes apparent that $\mathbf{V}_k^j$ for the nonholonomic system stands in parallel to $\partial\mathbf{R}_j/\partial q_k = \partial\dot{\mathbf{R}}_j/\partial\dot{q}_k$ for the holonomic system, and $\mathbf{V}_t$ has a similar parallel to $\partial\mathbf{R}/\partial t$. The $n$ vectors $\mathbf{V}_k^j$ are independent quantities, which can be substituted for $\partial\dot{\mathbf{R}}_j/\partial\dot{q}_k$ in Eq. (20) without sacrificing the feature of Eq. (20) as the source of $n$ independent differential equations. Thus the new equations of motion become (with $n$ replaced by $\nu$ in Eq. (12))

$$\sum_{j=1}^{N}(\mathbf{f}_j - m_j\ddot{\mathbf{R}}_j)\cdot\mathbf{V}_k^j = 0, \qquad k = 1, \cdots, n \tag{60a}$$

These equations are not sufficient to fully describe the system behavior, however, because in general they will involve all of the original generalized coordinates $q_1, \cdots, q_\nu$ and their first and second time derivatives. These equations of motion must therefore be augmented by the $m$ constraint equations appearing in Eq. (54) or Eq. (55). If none of the unknown constraint forces appearing in $\mathbf{f}_j$ in Eq. (60) survives the dot multiplications and summations in that equation, then they are classified as nonworking constraint forces, and Eqs. (60) and (54) combine to form a complete set, which is sufficient for the determination of $q_1, \cdots, q_\nu$.

Equation (60a) is in a convenient form for a multiple-particle system, but this equation is awkward for a system of rigid bodies or deformable continua. For each continuous body in the system, one can write the generic expression for the inertial velocity of a point (or differential element) of the body as $\dot{\mathbf{R}}$, and then define $\mathbf{V}_k$ from an expansion parallel to Eq. (59a), so that

$$\dot{\mathbf{R}} = \sum_{k=1}^{n} \mathbf{V}_k \dot{q}_k + \mathbf{V}_t, \qquad j = 1, \cdots, n \tag{59b}$$

Equation (60a) is then generalized to include continua by the addition to the $N$ particle sum of such terms as

$$\int \mathbf{V}_k \cdot (d\mathbf{f} - \ddot{\mathbf{R}}\, dm) \tag{60b}$$

for each deformable continuum. Although this expression could be expanded as in the previous sections, we will forego this labor, and concentrate on the special case of the rigid body.

Eq. (60a, b) is sometimes referred to as *Lagrange's form of D'Alembert's principle for simply constrained systems*; the development presented here follows Kane (Ref. 40).

For a rigid body, it is most convenient to express the generic inertial velocity $\dot{\mathbf{R}}$ and acceleration $\ddot{\mathbf{R}}$ in terms of the corresponding quantities $\dot{\mathbf{R}}^c$ and $\ddot{\mathbf{R}}^c$ for the mass center (see Fig. 3 in Subsection A-2). Then we have

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}^c + \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{61a}$$

and

$$\ddot{\mathbf{R}} = \ddot{\mathbf{R}}^c + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \tag{61b}$$

where $\boldsymbol{\omega}$ is the inertial angular velocity of the rigid body and $\boldsymbol{\rho}$ is the generic position vector from the mass center.

In parallel with Eq. (59a), we can define the quantities $\mathbf{V}_k^c$ and $\boldsymbol{\omega}_k$ by the expansions

$$\dot{\mathbf{R}}^c = \sum_{k=1}^{n} \mathbf{V}_k^c \dot{q}_k + \mathbf{V}_t^c \tag{62a}$$

and

$$\boldsymbol{\omega} = \sum_{k=1}^{n} \boldsymbol{\omega}_k \dot{q}_k + \boldsymbol{\omega}_t \tag{62b}$$

and then write $\dot{\mathbf{R}}$ as

$$\dot{\mathbf{R}} = \sum_{k=1}^{n} \mathbf{V}_k^c \dot{q}_k + \mathbf{V}_t^c + \left( \sum_{k=1}^{n} \boldsymbol{\omega}_k \dot{q}_k + \boldsymbol{\omega}_t \right) \times \boldsymbol{\rho} \tag{59c}$$

Comparison of Eqs. (59b, c) produces the identity

$$\mathbf{V}_k = \mathbf{V}_k^c + \boldsymbol{\omega}_k \times \boldsymbol{\rho} \tag{62c}$$

When this result is substituted into Eq. (60b), we find that each rigid body contributes to the equations of motion the quantity

$$\int \mathbf{V}_k \cdot (d\mathbf{f} - \ddot{\mathbf{R}} dm) = \int (\mathbf{V}_k^c + \boldsymbol{\omega}_k \times \boldsymbol{\rho}) \cdot \{d\mathbf{f} - [\ddot{\mathbf{R}}^c + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})]\, dm\}$$

$$= \mathbf{V}_k^c \cdot \int d\mathbf{f} - \mathbf{V}_k^c \cdot \ddot{\mathbf{R}}^c \int dm - \mathbf{V}_k^c \cdot \left\{ \dot{\boldsymbol{\omega}} \times \int \boldsymbol{\rho}\, dm + \boldsymbol{\omega} \times \left( \boldsymbol{\omega} \times \int \boldsymbol{\rho}\, dm \right) \right\}$$

$$+ \boldsymbol{\omega}_k \cdot \int \boldsymbol{\rho} \times d\mathbf{f} - \boldsymbol{\omega}_k \cdot \int \boldsymbol{\rho}\, dm \times \ddot{\mathbf{R}}^c$$

$$- \boldsymbol{\omega}_k \cdot \int \boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho})\, dm - \boldsymbol{\omega}_k \cdot \int \boldsymbol{\rho} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})]\, dm$$

By mass center definition we have $\int \boldsymbol{\rho}\, dm = 0$. If we incorporate the definitions

$$\mathbf{f} = \int d\mathbf{f}, \qquad \mathbf{m} \triangleq \int \boldsymbol{\rho} \times d\mathbf{f}$$

$$\mathcal{M} \triangleq \int dm, \qquad \mathbf{H} \triangleq \int \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})\, dm$$

we find that the preceding expression becomes

$$\int \mathbf{V}_k \cdot (d\mathbf{f} - \ddot{\mathbf{R}})\, dm = \mathbf{V}_k^c \cdot (\mathbf{f} - \mathcal{M}\ddot{\mathbf{R}}^c) + \boldsymbol{\omega}_k \cdot (\mathbf{m} - \dot{\mathbf{H}}) \tag{63}$$

(The expressions in parentheses we recognize to be zero from the Newton-Euler equations.)

Thus for a system of $\mathcal{P}$ particles and $\mathcal{B}$ extended rigid bodies, the equations of motion (60a) become

$$\sum_{j=1}^{\mathcal{P}} (\mathbf{f}_j - m_j \ddot{\mathbf{R}}_j) \cdot \mathbf{V}_k^j + \sum_{j=1}^{\mathcal{B}} [(\mathbf{f}^j - \mathcal{M}_j \ddot{\mathbf{R}}^j) \cdot \mathbf{V}_k^{c_j} + (\mathbf{m}^j - \dot{\mathbf{H}}^j) \cdot \boldsymbol{\omega}_k^j] = 0, \qquad k = 1, \cdots, n \tag{60c}$$

If these $n$ second order differential equations in the $\nu$ variables $q_1, \cdots, q_\nu$ are augmented by the $m$ constraint equations (Eq. (54)), and all unknown constraint forces classify as nonworking constraint forces, then the differential equation set is complete.

**4. Kane's quasi-coordinate formulation of D'Alembert's principle.** T. R. Kane has provided in Refs. 36 and 40 a generalization of the method of the previous section that often leads to substantial simplification of the equations of motion. Whereas Eq. (60a) of the previous section consists of the sum of $N$ dot products of $\mathbf{F}_j - m_j \ddot{\mathbf{R}}_j$ with quantities $\mathbf{V}_k^j$ as defined in Eq. (59a), we now write instead for the multiple-particle case the sum of $N$ dot products of $\mathbf{F}_j - m_j \ddot{\mathbf{R}}_j$ with a generalized definition of $\mathbf{V}_k^j$ as provided by

$$\dot{\mathbf{R}}_j \triangleq \sum_{k=1}^{n} \mathbf{V}_k^j u_k + \mathbf{V}_t^j, \qquad j = 1, \cdots, N \tag{64a}$$

with $u_1, \cdots, u_n$ defined by the nonsingular (invertible) relationship

$$u_k \triangleq \sum_{s=1}^{n} W_{sk} \dot{q}_s + w_k, \qquad k = 1, \cdots, n \tag{64b}$$

or its matrix counterpart

$$u = W^T \dot{q} + w \tag{64c}$$

with $u$ and $w$ defined as $n$ by 1 matrices, and $W$ a nonsingular $n$ by $n$ matrix. Evidently Eq. (59a) is the special case of Eq. (64a) corresponding to $W_{ks} = \delta_{ks}$ and $w_k = 0$ in Eq. (64b), or $W = U$ and $w = 0$ in Eq. (64c). Thus the definition of $\mathbf{V}_k^j$ implied by Eq. (64a) can stand as the general definition, subsuming that found in Eq. (59a).

The validity of Eq. (60a) for the more general definition of $\mathbf{V}_k^j$ is guaranteed by Eqs. (1) and (12), with $\nu$ replacing $n$ in the latter. Moreover, the new vectors $\mathbf{V}_k^j$ $(k = 1,\cdots,n)$ can remain an independent set, assuming that the $n$ quantities $u_1,\cdots,u_n$ in Eq. (64b) are each different linear combinations of the $n$ generalized velocities $\dot{q}_1,\cdots,\dot{q}_n$, and the latter are (as in the previous section) a subset of the original $\nu$ generalized velocities $\dot{q}_1,\cdots,\dot{q}_\nu$ obtained by imposing the constraint equations found in Eqs. (54) through (57).

The quantities $u_k$ $(k = 1,\cdots,n)$ in Eq. (64b) might in special instances be themselves recognizable as time derivatives of generalized coordinates; this is clearly the case when $W_{ks} \triangleq \delta_{ks}$ and $w_k \triangleq 0$ so that $u_k \triangleq \dot{q}_k$, but it follows also for any choice of constants for $W_{ks}$ and $w_k$, and for special functions of the generalized coordinates as well. More generally, however, the quantities $u_1,\cdots,u_n$ are not the time derivatives of any kinematical variables that qualify as generalized coordinates. The functions $u_1,\cdots,u_n$ are classically called *derivatives of quasi-coordinates* (see Ref. 39, p. 41 and Ref. 45, p. 197). The most notable examples of functions in this class are the scalar components of angular velocity in an orthogonal vector basis.

Following Kane (Ref. 40) for the multiple-particle case, we may define the scalars

$$f_k \triangleq \sum_{j=1}^{N} \mathbf{F}_j \cdot \mathbf{V}_k^j = \sum_{j=1}^{N} \mathbf{f}_j \cdot \mathbf{V}_k^j, \qquad k = 1,\cdots,n \tag{65a}$$

$$f_k^* \triangleq -\sum_{j=1}^{N} m_j \ddot{\mathbf{R}}_j \cdot \mathbf{V}_k^j, \qquad k = 1,\cdots,n \tag{65b}$$

and rewrite Eq. (60) in the form

$$f_k + f_k^* = 0, \qquad k = 1,\cdots,n \tag{66a}$$

The quantities $f_k$ and $f_k^*$ are generalizations of the quantities $Q_k$ and $Q_k^*$ defined for systems characterized by $n$ *independent* coordinates in Eq. (14). Kane applies to $f_k$ and $f_k^*$ the same names he has given us for $Q_k$ and $Q_k^*$; thus $f^k$ is called the $k$th *generalized active force* and $f_k^*$ is the $k$th *generalized inertia force*. Equation (66a) is thus a generalization of Eq. (15) with an important difference; even when unknown constraint forces are in the nonworking class, Eq. (66a) is not sufficient to solve for the unknowns $q_1,\cdots,q_\nu$ in the problem, and the constraint equations in Eq. (54) must be incorporated in the system of equations to be integrated simultaneously. Of course Eq. (66a) can be written as the matrix equation

$$f + f^{*} = 0 \tag{66b}$$

where $f \stackrel{\Delta}{=} \{f_1, \cdots, f_n\}^T$ and $f^{*} \stackrel{\Delta}{=} \{f_1^{*}, \cdots, f_n^{*}\}^T$. Then the constraint equations in the matrix form shown in Eq. (55) become appropriate.

Since we have constrained the definition of Kane's variables $u_1, \cdots, u_n$ in Eq. (64b) in such a way that one can always invert this equation to find

$$\dot{q}_j = \dot{q}_j(u_1, \cdots, u_n, q_1, \cdots, q_\nu, t), \qquad \text{for } j = 1, \cdots, n$$

we are assured of the possibility of writing $\dot{\mathbf{R}}_j$ in the form of Eq. (64a), and hence obtaining $\ddot{\mathbf{R}}_j = \ddot{\mathbf{R}}_j(\dot{u}_1, \cdots, \dot{u}_n, u_1, \cdots, u_n, \dot{q}_1, \cdots, \dot{q}_\nu, q_1, \cdots, q_\nu, t)$ for substitution into Eq. (65b). With this substitution the final equations of motion, Eq. (66a), become $n$ *first order* differential equations in the $n + \nu$ unknowns $u_1, \cdots, u_n, q_1, \cdots, q_\nu$. To obtain a complete formulation, the $n$ equations of motion (Eq. (66a)) must be combined with the $n$ kinematic equations (Eq. (64b)) which define Kane's variables and $m$ constraint equations in Eq. (54) or Eq. (55). Since $2n + m = n + \nu = 2\nu - m$, this combination provides a complete set of equations, repeated here as the matrix equations

$$f + f^{*} = 0 \tag{66b}$$

$$A\dot{q} + B = 0 \tag{55}$$

$$u = W^T \dot{q} + w \tag{64c}$$

Although these equations have been developed here only for a multiple-particle system, they are readily extended by the procedures of the previous section to apply to systems of $\mathcal{P}$ particles and $\mathcal{B}$ rigid bodies, or to nonrigid continua. By comparing Eqs. (60c) and (65) one can see that Kane's equations in this case are still represented by Eqs. (66b, 55, and 64c), with the substitution for Eqs. (65a, b) of the expressions

$$f_k \stackrel{\Delta}{=} \sum_{j=1}^{\mathcal{P}} \mathbf{f}_j \cdot \mathbf{V}_k^j + \sum_{j=1}^{\mathcal{B}} [\mathbf{f}^j \cdot \mathbf{V}_k^{c_j} + \mathbf{m}^j \cdot \boldsymbol{\omega}_k^j] \tag{65c}$$

and

$$f_k^{*} \stackrel{\Delta}{=} -\sum_{j=1}^{\mathcal{P}} m_j \ddot{\mathbf{R}}_j \cdot \mathbf{V}_k^j - \sum_{j=1}^{\mathcal{B}} [\mathcal{M}_j \ddot{\mathbf{R}}^j \cdot \mathbf{V}_k^{c_j} + \dot{\mathbf{H}}^j \cdot \boldsymbol{\omega}_k^j] \tag{65d}$$

In these equations the vectors $\mathbf{V}_k^{c_j}$ and $\boldsymbol{\omega}_k^j$ are obtained from the following generalization of Eqs. (62), following the pattern of Eq. (64a):

$$\dot{\mathbf{R}}^j \stackrel{\Delta}{=} \sum_{k=1}^{n} \mathbf{V}_k^{c_j} u_k + \mathbf{V}_t^{c_j}, \qquad j = 1, \cdots, \mathcal{B} \tag{65e}$$

$$\boldsymbol{\omega}^j \stackrel{\Delta}{=} \sum_{k=1}^{n} \boldsymbol{\omega}_k^j u_k + \boldsymbol{\omega}_t^j, \qquad j = 1, \cdots, \mathcal{B} \tag{65f}$$

As presented in Refs. 36 and 40, Kane's method is limited in application to particles and rigid bodies, but extension to deformable continua is straightforward if deformations are limited to those that can be characterized by a finite number of generalized coordinates, $q_1, \cdots, q_\nu$. In this case Eq. (66a) still applies if $f_k$ and $f_k^*$ are defined by

$$f_k \overset{\Delta}{=} \int \mathbf{V}_k \cdot d\mathbf{f} \tag{65g}$$

$$f_k^* \overset{\Delta}{=} -\int \mathbf{V}_k \cdot \ddot{\mathbf{R}}\, dm \tag{65h}$$

where $\mathbf{V}_k$ is defined by

$$\dot{\mathbf{R}} = \sum_{k=1}^{n} \mathbf{V}_k u_k + \mathbf{V}_t \tag{65i}$$

and $u_k$ is defined by Eq. (64).

Equations (65g, h) for Kane's method are parellel to Eqs. (32a, b) for Lagrange's form of D'Alembert's principle. Just as the latter were manipulated to obtain more explicit results in the form of Eq. (41), so also Eqs. (65g, h) would have to be manipulated to obtain from their sum an appropriate final form of the equations of motion.

## B. Lagrange's Equations

**1. Lagrange's equations for independent generalized coordinates.** In combination, Eqs. (19a) and (20) become

$$\sum_{j=1}^{N} m_j \ddot{\mathbf{R}}_j \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k} = Q_k, \qquad k = 1, \cdots, n \tag{67}$$

Equation (67) provides yet another statement of Lagrange's form of D'Alembert's principle for holonomic systems; the resulting set of equations is complete if all unknown constraint forces are of the "nonworking" class defined by Eq. (12). To apply these equations directly, one must undertake the chore of calculating $\ddot{\mathbf{R}}_j$ for each particle in the system. Lagrange observed that he could replace the left side of Eq. (67) with a somewhat simpler expression involving the *kinetic energy*, as defined for a system of $N$ particles by

$$T \overset{\Delta}{=} \frac{1}{2} \sum_{j=1}^{N} m_j \dot{\mathbf{R}}_j \cdot \dot{\mathbf{R}}_j \tag{68}$$

This step can be accomplished with the expansion

$$\begin{aligned} \sum_{j=1}^{N} m_j \ddot{\mathbf{R}}_j \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k} &= \frac{d}{dt}\left(\sum_{j=1}^{N} m_j \dot{\mathbf{R}}_j \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k}\right) - \sum_{j=1}^{N} m_j \dot{\mathbf{R}}_j \cdot \frac{d}{dt}\left(\frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k}\right) \\ &= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}\sum_{j=1}^{N} m_j \dot{\mathbf{R}}_j \cdot \dot{\mathbf{R}}_j\right)\right] - \sum_{j=1}^{N} m_j \dot{\mathbf{R}}_j \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{R}_j}{\partial q_k}\right) \end{aligned}$$

where in the last step we have utilized Eq. (17). Substituting Eq. (68) into this equation and reversing the differentiation sequence in the final term provides

$$\sum_{j=1}^{N} m_j \ddot{\mathbf{R}}_j \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial \dot{q}_k} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \sum_{j=1}^{N} m_j \dot{\mathbf{R}}_j \cdot \frac{\partial \dot{\mathbf{R}}_j}{\partial q_k}$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial}{\partial q_k}\left(\frac{1}{2}\sum_{j=1}^{N} m_j \dot{\mathbf{R}}_j \cdot \dot{\mathbf{R}}_j\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} \tag{69}$$

Thus Eq. (67) adopts the familiar form of *Lagrange's equations for independent generalized coordinates.*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k, \qquad k = 1, \cdots, n \tag{70a}$$

In matrix form, with the conventions of Appendix A, Eq. (70a) becomes

$$\frac{d}{dt}(T_{,\dot{q}}) - T_{,q} = Q \tag{70b}$$

If the generalized force matrix $Q$ contains only *conservative* forces, then by definition it can be written as

$$Q = -\partial V / \partial q$$

where $V$ is a scalar called the potential energy. For systems in this class it is customary to define the Lagrangian $\mathcal{L} \triangleq T - V$ and to write Eq. (70b) as

$$\frac{d}{dt}(\mathcal{L}_{,\dot{q}}) - \mathcal{L}_{,q} = 0 \tag{70c}$$

Having obtained Eq. (70c) as a representation of the equations of motion of a given system in terms of the *independent* generalized coordinates $q_1, \cdots, q_n$ in the matrix $q$, we can now observe that if we make a transformation from $q$ to a larger set of $\nu$ redundant coordinates in the matrix $q'$, and transform $\mathcal{L}(q, \dot{q}, t)$ to $\mathcal{L}'(q', \dot{q}', t)$, then the $\nu$ scalar equations of motion in the redundant coordinates can be written (see Ref. 43)

$$\frac{d}{dt}(\mathcal{L}'_{,\dot{q}'}) - \mathcal{L}'_{,q'} = 0 \tag{70d}$$

This equation must be augmented by constraint equations if we are to obtain a complete set.

The generality of this result is limited by the hypothesis that there exists in the first place a set of independent generalized coordinates. The procedure in Subsection B-2 will not be so restricted.

*a. Kinetic energy expressions.* The utility of Eq. (70) in comparison with Lagrange's form of D'Alembert's principle depends upon the difficulty encountered in calculating the kinetic energy $T$. The definition of $T$ in Eq. (68) is in a convenient form for a system of particles, but more compact representations are more useful for systems of particles and rigid bodies and for continuous distributions of mass. As noted in the footnote following Eq. (1), we will not hesitate to apply Eq. (70) to continuous distributions of matter, even though strictly speaking we derived Eq. (70) only for a system of particles finite in number. In application to a continuum, we replace Eq. (68) by

$$T \triangleq \frac{1}{2}\int \dot{\mathbf{R}} \cdot \dot{\mathbf{R}}\, dm \tag{71}$$

where the integration extends over a material system of constant mass and $\mathbf{R}$ is the position vector locating the differential mass $dm$ relative to an inertially fixed point.

When the physical system is idealized as a collection of $\mathcal{P}$ particles and $\mathcal{B}$ rigid bodies, it is convenient to use Eq. (71) for each rigid body, treating it as a rigid continuum. Then if $\dot{\mathbf{R}}^p$ is the inertial velocity of a point $p$ fixed in the body, and $\mathbf{r}$ is the generic position vector from $p$ to a field point of the body, we have for each rigid body the kinetic energy

$$\begin{aligned} T &= \frac{1}{2}\int \dot{\mathbf{R}} \cdot \dot{\mathbf{R}}\, dm \\ &= \frac{1}{2}\int (\dot{\mathbf{R}}^p + \dot{\mathbf{r}}) \cdot (\dot{\mathbf{R}}^p + \dot{\mathbf{r}})\, dm \\ &= \frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^p \cdot \dot{\mathbf{R}}^p + \dot{\mathbf{R}}^p \cdot \int \dot{\mathbf{r}}\, dm + \frac{1}{2}\int \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\, dm \\ &= \frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^p \cdot \dot{\mathbf{R}}^p + \dot{\mathbf{R}}^p \cdot \boldsymbol{\omega} \times \int \mathbf{r}\, dm + \frac{1}{2}\int (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r})\, dm \end{aligned}$$

where $\mathcal{M}$ is the mass of the rigid body. The integrand of the final term can be expanded as

$$\begin{aligned} (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r}) &= \boldsymbol{\omega} \cdot \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= \boldsymbol{\omega} \cdot [\mathbf{r} \cdot \mathbf{r}\boldsymbol{\omega} - \mathbf{r}\mathbf{r} \cdot \boldsymbol{\omega}] = \boldsymbol{\omega} \cdot [\mathbf{r} \cdot \mathbf{r}\, \mathbf{U} - \mathbf{r}\mathbf{r}] \cdot \boldsymbol{\omega} \end{aligned}$$

where $\mathbf{U}$ is the *unit dyadic*. If now we introduce the *inertia dyadic* for point $p$ as

$$\mathbf{J}^p \triangleq \int (\mathbf{r} \cdot \mathbf{r}\, \mathbf{U} - \mathbf{r}\mathbf{r})\, dm \tag{72}$$

and use the mass center definition to write

$$\int \mathbf{r}\, dm = \mathcal{M}\mathbf{r}^c \tag{73}$$

where $\mathbf{r}^c$ is the position vector from $p$ to the mass center $c$ of the rigid body, then we have

$$T = \frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^p \cdot \dot{\mathbf{R}}^p + \mathcal{M}\dot{\mathbf{R}}^p \cdot \boldsymbol{\omega} \times \mathbf{r}^c + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{J}^p \cdot \boldsymbol{\omega} \tag{74}$$

Thus for a system of $\mathscr{P}$ particles and $\mathscr{B}$ rigid bodies, the kinetic energy is given by

$$T = \sum_{j=1}^{\mathscr{P}} \frac{1}{2} m_j \dot{\mathbf{R}}_j \cdot \dot{\mathbf{R}}_j + \sum_{j=1}^{\mathscr{B}} \left[ \frac{1}{2} \mathscr{M}_j \dot{\mathbf{R}}^{p_j} \cdot \dot{\mathbf{R}}^{p_j} + \mathscr{M}_j \dot{\mathbf{R}}^{p_j} \cdot \boldsymbol{\omega}^j \times \mathbf{r}^{c_j} + \frac{1}{2} \boldsymbol{\omega}^j \cdot \mathbf{J}^{p_j} \cdot \boldsymbol{\omega}^j \right] \tag{75}$$

Finally, the physical system might be idealized as a set of $\mathscr{P}$ particles and $\mathscr{B}$ extended bodies, of which the subset $\bar{\mathscr{B}}$ is deformable. Then for each deformable body we can introduce a reference frame $f$ in which a point $p$ remains fixed, and write for substitution into Eq. (71)

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}^p + \frac{d}{dt}(\mathbf{r} + \mathbf{u})$$

where $\mathbf{r}$ is fixed in the frame in question. If now $\mathring{\mathbf{u}}$ is introduced to symbolize the time derivative of $\mathbf{u}$ in this reference frame $f$, we have

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}^p + \boldsymbol{\omega} \times (\mathbf{r} + \mathbf{u}) + \mathring{\mathbf{u}}$$

and for a deformable body

$$\begin{aligned} T = {} & \frac{1}{2} \mathscr{M} \dot{\mathbf{R}}^p \cdot \dot{\mathbf{R}}^p + \mathscr{M} \dot{\mathbf{R}}^p \cdot (\boldsymbol{\omega} \times \mathbf{r}^c) + \dot{\mathbf{R}}^p \cdot \left( \boldsymbol{\omega} \times \int \mathbf{u}\, dm \right) \\ & + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J}^p \cdot \boldsymbol{\omega} + \dot{\mathbf{R}}^p \cdot \int \mathring{\mathbf{u}}\, dm + \boldsymbol{\omega} \cdot \int (\mathbf{r} + \mathbf{u}) \times \mathring{\mathbf{u}}\, dm \\ & + \frac{1}{2} \int \mathring{\mathbf{u}} \cdot \mathring{\mathbf{u}}\, dm \end{aligned} \tag{76}$$

with

$$\mathbf{J}^p \triangleq \int [(\mathbf{r} + \mathbf{u}) \cdot (\mathbf{r} + \mathbf{u}) \mathbf{U} - (\mathbf{r} + \mathbf{u})(\mathbf{r} + \mathbf{u})]\, dm \tag{77}$$

representing the time-varying inertia dyadic of the deformable body referred to $p$.

At this point we have not yet established a unique identity for the floating reference frame $f$ we have been using in our expression for the kinetic energy of a deformable body; we have simply required that $p$ and $\mathbf{r}$ be fixed in $f$, and this constraint leaves open a myriad of possibilities, each with its own $p$ and $\mathbf{r}$-vectors. The same issue arose in the earlier discussion of continuum dynamics (see Subsection A-2), and in that context it was noted that particular choices of this reference frame could be made to simplify the expression for the angular momentum $\mathbf{H}$ in Eq. (43); now we can see that these choices also simplify $T$, and correspondingly reduce the complexity of the equations of motion appearing as Eq. (70). Specifically, if we let $p$ be the mass center $c$ of the deformable body, the vector $\mathbf{r}^c$ and the integral $\int \mathbf{u}\, dm$ disappear from $T$ in Eq. (76). We can further restrict the reference frame by requiring that it rotate in such a way that $\int \mathbf{r} \times \mathbf{u}\, dm = 0$. The reference frame meeting these restrictions we have called the *Buckens frame* $f_B$. With these restrictions, the kinetic energy in Eq. (76) becomes

$$T = \frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^c \cdot \dot{\mathbf{R}}^c + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \int \mathbf{u} \times \mathring{\mathbf{u}}\, dm + \frac{1}{2}\int \mathring{\mathbf{u}} \cdot \mathring{\mathbf{u}}\, dm \tag{78}$$

where $\mathbf{I}$ is the time-varying inertia dyadic for the deformable body referred to its mass center. This equation must be accompanied by the constraint equations

$$\int \mathbf{u}\, dm = 0, \qquad \int \mathbf{r} \times \mathbf{u}\, dm = 0 \tag{79}$$

which were previously presented as Eqs. (37) and (47).

As noted in Subsection A-2, the selection of the Buckens frame as the floating reference frame $f$ with respect to which deformations $\mathbf{u}$ are to be measured is only one of a variety of reasonable choices. One may instead choose to work with the *Tisserand frame*, $f_T$, with respect to which the mass center is fixed and the angular momentum is zero. Then the constraint equations become the more complex relations

$$\int \mathbf{u}\, dm = 0$$
$$\int (\mathbf{r} + \mathbf{u}) \times \mathring{\mathbf{u}}\, dm = 0 \tag{80}$$

(previously presented as Eqs. (37) and (44)), and the kinetic energy simplifies to

$$T = \frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^c \cdot \dot{\mathbf{R}}^c + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} + \frac{1}{2}\int \mathring{\mathbf{u}} \cdot \mathring{\mathbf{u}}\, dm \tag{81}$$

Yet another possibility is to choose the principal axis frame $f_P$, such that the mass center and the principal axes of the deforming body remain fixed in it. Then if $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are orthogonal unit vectors fixed in $f_P$ and aligned with the mass center principal axes of the body when undeformed, we have the constraint equations (see Eqs. (37) and (50a))

$$\int \mathbf{u}\, dm = 0, \qquad \mathbf{p}_1 \cdot \mathbf{I} \cdot \mathbf{p}_2 = \mathbf{p}_1 \cdot \mathbf{I} \cdot \mathbf{p}_3 = \mathbf{p}_2 \cdot \mathbf{I} \cdot \mathbf{p}_3 = 0 \tag{82}$$

The kinetic energy in this case would become

$$T = \frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^c \cdot \dot{\mathbf{R}}^c + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \int (\mathbf{r} + \mathbf{u}) \times \mathring{\mathbf{u}}\, dm + \frac{1}{2}\int_j \mathring{\mathbf{u}} \cdot \mathring{\mathbf{u}}\, dm \tag{83}$$

The advantage of this choice stems from the fact that one knows in advance that the inertia matrix $I$ representing $\mathbf{I}$ in the $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ vector basis is diagonal.

Since there may be advantages in any of these choices of the floating reference frame (and perhaps also in other options, including the possibility of fixing the reference point $p$ at some connection point or other point of the physical system), we shall make no commitment here, retaining the general form for $T$ provided in Eq. (76). If now we return to the consideration of a physical system idealized as a collection of $\mathcal{P}$ particles and $\mathcal{B}$ extended bodies, of which the subset $\bar{\mathcal{B}}$ are deformable bodies, we obtain the system kinetic energy as

$$
\begin{aligned}
T = {} & \sum_{j=1}^{\mathcal{P}} \frac{1}{2} m_j \dot{\mathbf{R}}_j \cdot \dot{\mathbf{R}}_j + \sum_{j=1}^{\mathcal{B}} \left\{ \frac{1}{2} \mathcal{M}_j \dot{\mathbf{R}}^{p_j} \cdot \dot{\mathbf{R}}^{p_j} + \mathcal{M}_j \dot{\mathbf{R}}^{p_j} \cdot \boldsymbol{\omega}^j \times \mathbf{r}^{c_j} + \frac{1}{2} \boldsymbol{\omega}^j \cdot \mathbf{J}^{p_j} \cdot \boldsymbol{\omega}^j \right\} \\
& + \sum_{j=1}^{\bar{\mathcal{B}}} \left\{ \dot{\mathbf{R}}^{p_j} \cdot \boldsymbol{\omega}^j \times \int_j \mathbf{u}\, dm + \dot{\mathbf{R}}^{p_j} \cdot \int_j \mathring{\mathbf{u}}\, dm + \boldsymbol{\omega}^j \cdot \int_j (\mathbf{r} + \mathbf{u}) \times \mathring{\mathbf{u}}\, dm \right. \\
& \left. + \frac{1}{2} \int_j \mathring{\mathbf{u}} \cdot \mathring{\mathbf{u}}\, dm \right\}
\end{aligned}
\tag{84}
$$

*b. Relationship between Lagrange's equations and Newton's particle equations.* In attempting to compare the utility of Lagrange's equations for independent generalized coordinates to that of Lagrange's form of D'Alembert's principle, we must recognize that the link between them is very direct; the journey from Eq. (67) to Eq. (70) involves nothing more than a restatement of the left side of Eq. (67) in terms of the kinetic energy. Thus Eqs. (67) and (70) differ in appearance only, and must give identical results. Because Eq. (70) involves only scalars, and requires only enough vectorial kinematics to calculate inertial velocities (rather than accelerations), this is usually considered the simpler form for an analyst to work with. The advantage would also lie with Eq. (70) for the programmer faced with the task of teaching a digital computer to obtain literal (nonnumerical) equations of motion using a symbolic manipulation code (such as FORMAC) and operating on a given expression for the scalar $T$. In summary we can conclude that for a system with independent generalized coordinates it is probably easier to *obtain* equations of motion with Lagrange's equations than with Lagrange's form of D'Alembert's principle; but the resulting equations are identical, so neither has any advantage in solution efficiency.

The point made in the preceding paragraph is rather obvious for a system of $N$ particles, since the transition from Eq. (67) to Eq. (70) is so direct. The equivalence of Eqs. (67) and (70) for a system of particles implies the equivalence of Eqs. (28) and (70) for a system of particles and rigid bodies, at least as long as we accept the definition of a rigid body as a finite set of particles with fixed distances among them; this equivalence establishes the important relationship between the Newton-Euler equations of motion and Lagrange's equations. Because this relationship is important to the method-comparisons that lie ahead in this report, it seems worthwhile to establish the link between Eqs. (28) and (70) directly by deriving the latter from the Newton-Euler equations, and observing that Eq. (28) occupies the middle ground between them. It will suffice here to achieve this derivation for a single rigid body, since generalization to a system of $\mathcal{P}$ particles and $\mathcal{B}$ rigid bodies is straightforward but notationally complex.

*c. Derivation from Newton-Euler equations for a rigid body.* We begin with Newton's second law and its rotational consequence for a rigid body, as expressed by Euler:

$$\mathbf{F} = \mathcal{M}\ddot{\mathbf{R}}^c \tag{85}$$

$$\mathbf{M} = \dot{\mathbf{H}} \tag{86}$$

where (as previously) $\mathbf{F}$ is the resultant force on the body, $\mathcal{M}$ is the mass, $\mathbf{R}^c$ is the inertial position vector of the mass center $c$, and $\mathbf{M}$ and $\mathbf{H}$ are respectively resultant moment and angular momentum referred to $c$.

If the rigid body motion is described in terms of $n$ independent generalized coordinates, one can obtain $n$ independent scalar equations from Eqs. (85) and (86) as follows:

$$\frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_\alpha}\cdot \mathbf{F} + \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_\alpha}\cdot \mathbf{M} = \frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_\alpha}\cdot \mathcal{M}\ddot{\mathbf{R}}^c + \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_\alpha}\cdot \dot{\mathbf{H}}, \qquad \alpha = 1, \cdots, n \tag{87}$$

This result is evidently a special case of Eq. (28), representing Lagrange's form of D'Alembert's principle; there remains the task of proving Eq. (87) to be equivalent to Eq. (70), representing Lagrange's equations. The equivalence of the left side of Eq. (87) and the right side of Eq. (70) is immediate, since we have in Eq. (22) already written the generalized force in this form (see Eqs. (12) and (23)).

It is also easy to show the identity

$$\begin{aligned}
\frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_\alpha}\cdot \mathcal{M}\ddot{\mathbf{R}}^c &= \frac{d}{dt}\left[\frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_\alpha}\cdot \mathcal{M}\dot{\mathbf{R}}^c\right] - \left[\frac{d}{dt}\left(\frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_\alpha}\right)\right]\cdot \mathcal{M}\dot{\mathbf{R}}^c \\
&= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_\alpha}\left(\frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^c\cdot \dot{\mathbf{R}}^c\right] - \left[\frac{d}{dt}\left(\frac{\partial \mathbf{R}^c}{\partial q_\alpha}\right)\right]\cdot \mathcal{M}\dot{\mathbf{R}}^c \\
&= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_\alpha}\left(\frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^c\cdot \dot{\mathbf{R}}^c\right)\right] - \frac{\partial \dot{\mathbf{R}}^c}{\partial q_\alpha}\cdot \mathcal{M}\dot{\mathbf{R}}^c \\
&= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_\alpha}\left(\frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^c\cdot \dot{\mathbf{R}}^c\right)\right] - \frac{\partial}{\partial q_\alpha}\left(\frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^c\cdot \dot{\mathbf{R}}^c\right)
\end{aligned}$$

From Eq. (78) we can recognize that if we write the kinetic energy $T$ of a rigid body as

$$T = \frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^c\cdot \dot{\mathbf{R}}^c + \frac{1}{2}\boldsymbol{\omega}\cdot \mathbf{I}\cdot \boldsymbol{\omega} \stackrel{\Delta}{=} T_t + T_r$$

designating $T_t$ and $T_r$ respectively as the *translational kinetic energy* and the *rotational kinetic energy*, then the preceding equation becomes

$$\frac{\partial \dot{\mathbf{R}}^c}{\partial \dot{q}_\alpha}\cdot \mathcal{M}\ddot{\mathbf{R}}^c = \frac{d}{dt}\left(\frac{\partial T_t}{\partial \dot{q}_\alpha}\right) - \frac{\partial T_t}{\partial q_\alpha} \tag{88}$$

Thus we have established that Eq. (87) is equivalent to

$$Q_\alpha = \frac{d}{dt}\left(\frac{\partial T_t}{\partial \dot{q}_\alpha}\right) - \frac{\partial T_t}{\partial q_\alpha} + \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_\alpha}\cdot \dot{\mathbf{H}} \tag{89}$$

In order to establish that Eq. (87) and (70) are identical we need only establish the identity

$$\frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_\alpha}\cdot \dot{\mathbf{H}} = \frac{d}{dt}\left(\frac{\partial T_r}{\partial \dot{q}_\alpha}\right) - \frac{\partial T_r}{\partial q_\alpha} \tag{90}$$

But Eq. (90) is not so easily proven.

The required proof is facilitated by the identity

$$\begin{aligned}\frac{d}{dt}T_r &= \frac{d}{dt}\left(\frac{1}{2}\,\omega\cdot\mathbf{H}\right) = \frac{1}{2}\dot{\mathbf{H}}\cdot\omega + \frac{1}{2}\mathbf{H}\cdot\dot{\omega} = \frac{1}{2}\dot{\mathbf{H}}\cdot\omega + \frac{1}{2}\,\omega\cdot\mathsf{I}\cdot\dot{\omega}\\ &= \frac{1}{2}\dot{\mathbf{H}}\cdot\omega + \frac{1}{2}\omega\cdot\dot{\mathbf{H}} - \frac{1}{2}\omega\cdot\dot{\mathsf{I}}\cdot\omega = \omega\cdot\dot{\mathbf{H}} - \frac{1}{2}\omega\cdot[\omega\times\mathsf{I} - \mathsf{I}\times\omega]\cdot\omega\\ &= \omega\cdot\dot{\mathbf{H}}\end{aligned} \tag{91}$$

Thus the left side of Eq. (90) is

$$\dot{\mathbf{H}}\cdot\frac{\partial\omega}{\partial\dot{q}_\alpha} = \frac{\partial}{\partial\dot{q}_\alpha}(\dot{\mathbf{H}}\cdot\omega) - \frac{\partial\dot{\mathbf{H}}}{\partial\dot{q}_\alpha}\cdot\omega = \frac{\partial}{\partial\dot{q}_\alpha}\frac{dT_r}{dt} - \frac{\partial\dot{\mathbf{H}}}{\partial\dot{q}_\alpha}\cdot\omega$$

But the "chain rule" of differentiation provides

$$\frac{dT_r}{dt} = \sum_{\gamma=1}^{n}\left(\frac{\partial T_r}{\partial\dot{q}_\gamma}\ddot{q}_\gamma + \frac{\partial T_r}{\partial q_\gamma}\dot{q}_\gamma\right) + \frac{\partial T_r}{\partial t} \tag{92}$$

so that

$$\begin{aligned}\frac{\partial}{\partial\dot{q}_\alpha}\frac{dT_r}{dt} &= \frac{\partial T_r}{\partial q_\alpha} + \sum_{\gamma=1}^{n}\left(\frac{\partial^2 T_r}{\partial\dot{q}_\gamma\partial\dot{q}_\alpha}\ddot{q}_\gamma + \frac{\partial^2 T_r}{\partial q_\gamma\partial\dot{q}_\alpha}\dot{q}_\gamma\right) + \frac{\partial^2 T_r}{\partial t\,\partial\dot{q}_\alpha}\\ &= \frac{\partial T_r}{\partial q_\alpha} + \frac{d}{dt}\left(\frac{\partial T_r}{\partial\dot{q}_\alpha}\right)\end{aligned} \tag{93}$$

Therefore we have

$$\dot{\mathbf{H}}\cdot\frac{\partial\omega}{\partial\dot{q}_\alpha} = \frac{d}{dt}\frac{\partial T_r}{\partial\dot{q}_\alpha} + \frac{\partial T_r}{\partial q_\alpha} - \frac{\partial\dot{\mathbf{H}}}{\partial\dot{q}_\alpha}\cdot\omega \tag{94}$$

But, as in Eq. (93),

$$\frac{\partial}{\partial\dot{q}_\alpha}\frac{d\mathbf{H}}{dt} = \frac{\partial\mathbf{H}}{\partial q_\alpha} + \frac{d}{dt}\left(\frac{\partial\mathbf{H}}{\partial\dot{q}_\alpha}\right) \tag{95}$$

Therefore

$$\dot{\mathbf{H}}\cdot\frac{\partial\omega}{\partial\dot{q}_\alpha} = \frac{d}{dt}\frac{\partial T_r}{\partial\dot{q}_\alpha} + \frac{\partial T_r}{\partial q_\alpha} - \frac{\partial\mathbf{H}}{\partial q_\alpha}\cdot\omega - \left[\frac{d}{dt}\frac{\partial\mathbf{H}}{\partial\dot{q}_\alpha}\right]\cdot\omega$$

But

$$\frac{\partial\mathbf{H}}{\partial q_\alpha}\cdot\omega = \left[\frac{\partial}{\partial q_\alpha}(\omega\cdot\mathsf{I})\right]\cdot\omega = \frac{\partial}{\partial q_\alpha}\left(\frac{2}{2}\,\omega\cdot\mathsf{I}\cdot\omega\right) - \mathbf{H}\cdot\frac{\partial\omega}{\partial q_\alpha} = 2\frac{\partial T_r}{\partial q_\alpha} - \mathbf{H}\cdot\frac{\partial\omega}{\partial q_\alpha}$$

So we have

$$\dot{\mathbf{H}}\cdot\frac{\partial\omega}{\partial\dot{q}_\alpha} = \frac{d}{dt}\left(\frac{\partial T_r}{\partial\dot{q}_\alpha}\right) - \frac{\partial T_r}{\partial q_\alpha} - \left[\frac{d}{dt}\frac{\partial\mathbf{H}}{\partial\dot{q}_\alpha}\right]\cdot\omega + \mathbf{H}\cdot\frac{\partial\omega}{\partial q_\alpha} \tag{96}$$

Comparison of Eqs. (96) and (90) indicates that both are correct only if

$$\mathbf{H}\cdot\frac{\partial\omega}{\partial q_\alpha} - \left[\frac{d}{dt}\frac{\partial\mathbf{H}}{\partial\dot{q}_\alpha}\right]\cdot\omega = 0 \tag{97}$$

To prove Eq. (97), note that

$$
\begin{aligned}
\mathbf{H}\cdot\frac{\partial\omega}{\partial q_\alpha} - \left[\frac{d}{dt}\left(\frac{\partial\mathbf{H}}{\partial\dot{q}_\alpha}\right)\right]\cdot\omega &= \omega\cdot\mathsf{I}\cdot\frac{\partial\omega}{\partial q_\alpha} - \omega\cdot\frac{d}{dt}\left(\frac{\partial\mathsf{I}\cdot\omega}{\partial\dot{q}_\alpha}\right)\\
&= \omega\cdot\mathsf{I}\cdot\frac{\partial\omega}{\partial q_\alpha} - \omega\cdot\frac{d}{dt}\left(\mathsf{I}\cdot\frac{\partial\omega}{\partial\dot{q}_\alpha}\right)\\
&= \omega\cdot\mathsf{I}\cdot\frac{\partial\omega}{\partial q_\alpha} - \omega\cdot\left[\left(\frac{d\mathsf{I}}{dt}\right)\cdot\frac{\partial\omega}{\partial\dot{q}_\alpha} + \mathsf{I}\cdot\frac{d}{dt}\left(\frac{\partial\omega}{\partial\dot{q}_\alpha}\right)\right]\\
&= \omega\cdot\mathsf{I}\cdot\frac{\partial\omega}{\partial q_\alpha} - \omega\cdot\left[\omega\times\mathsf{I}\cdot\frac{\partial\omega}{\partial\dot{q}_\alpha} - \mathsf{I}\times\omega\cdot\frac{\partial\omega}{\partial\dot{q}_\alpha}\right.\\
&\qquad \left. + \mathsf{I}\cdot\frac{d}{dt}\left(\frac{\partial\omega}{\partial\dot{q}_\alpha}\right)\right]\\
&= \omega\cdot\mathsf{I}\cdot\frac{\partial\omega}{\partial q_\alpha} + \omega\cdot\mathsf{I}\times\omega\cdot\frac{\partial\omega}{\partial\dot{q}_\alpha} - \omega\cdot\mathsf{I}\cdot\frac{d}{dt}\left(\frac{\partial\omega}{\partial\dot{q}_\alpha}\right)\\
&= \omega\cdot\mathsf{I}\cdot\frac{\partial\omega}{\partial q_\alpha} + \omega\cdot\mathsf{I}\cdot\omega\times\frac{\partial\omega}{\partial\dot{q}_\alpha} - \omega\cdot\mathsf{I}\cdot\frac{d}{dt}\left(\frac{\partial\omega}{\partial\dot{q}_\alpha}\right)\\
&= \omega\cdot\mathsf{I}\cdot\left[\frac{\partial\omega}{\partial q_\alpha} + \omega\times\frac{\partial\omega}{\partial\dot{q}_\alpha} - \frac{d}{dt}\frac{\partial\omega}{\partial\dot{q}_\alpha}\right] \qquad (98)
\end{aligned}
$$

The right side of Eq. (98) is zero because of the purely kinematical theorem[4]

$$\frac{\partial\omega}{\partial q_\alpha} + \omega\times\frac{\partial\omega}{\partial\dot{q}_\alpha} - \frac{d}{dt}\left(\frac{\partial\omega}{\partial\dot{q}_\alpha}\right) = 0 \qquad (99)$$

Proof of this valuable theorem follows most directly from the definition of angular velocity $\omega$ given in Ref. 40, page 21: if $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ comprise a dextral, orthogonal set of unit vectors fixed in body $b$, then the inertial angular velocity of $b$ is given by

$$\omega = \dot{\mathbf{b}}_2\cdot\mathbf{b}_3\mathbf{b}_1 + \dot{\mathbf{b}}_3\cdot\mathbf{b}_1\mathbf{b}_2 + \dot{\mathbf{b}}_1\cdot\mathbf{b}_2\mathbf{b}_3 \qquad (100)$$

where dot means inertial time derivative, so that

$$\dot{\mathbf{b}}_i = \omega\times\mathbf{b}_i, \qquad i = 1,\cdots,n \qquad (101)$$

Since $\mathbf{b}_i = \mathbf{b}_i\,(q_1,\cdots,q_n,t)$, we can also expand $\dot{\mathbf{b}}_i$ as

$$\dot{\mathbf{b}}_i = \sum_{\alpha=1}^{n}\frac{\partial\mathbf{b}_i}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial\mathbf{b}_i}{\partial t}$$

thereby establishing the identities

$$\frac{\partial\dot{\mathbf{b}}_i}{\partial\dot{q}_\alpha} = \frac{\partial\mathbf{b}_i}{\partial q_\alpha} \qquad \text{and} \qquad \frac{\partial\mathbf{b}_i}{\partial\dot{q}_\alpha} = 0 \qquad (102)$$

[4]Professor T. R. Kane of Stanford University provided the proof of this theorem in personal correspondence, and thereby accomplished the key step in the proof of Eq. (90), permitting direct demonstration of the equivalence of Eqs. (87) and (70).

In the following proof of Eq. (99), the subscript $\alpha$ is dropped from $q_\alpha$ and the comma convention is used to represent partial derivatives, as in Appendix A. The left side of Eq. (99) then becomes

$$
\begin{aligned}
\boldsymbol{\omega}_{,q} + \boldsymbol{\omega} \times \boldsymbol{\omega}_{,\dot{q}} - \frac{d}{dt}\boldsymbol{\omega}_{,\dot{q}} = {} & (\dot{\mathbf{b}}_2 \cdot \mathbf{b}_3\mathbf{b}_1 + \dot{\mathbf{b}}_3 \cdot \mathbf{b}_1\mathbf{b}_2 + \dot{\mathbf{b}}_1 \cdot \mathbf{b}_2\mathbf{b}_3)_{,q} \\
& - \frac{d}{dt}\left(\dot{\mathbf{b}}_{2,\dot{q}} \cdot \mathbf{b}_3\mathbf{b}_1 + \dot{\mathbf{b}}_{3,\dot{q}} \cdot \mathbf{b}_1\mathbf{b}_2 + \dot{\mathbf{b}}_{1,\dot{q}} \cdot \mathbf{b}_2\mathbf{b}_3\right) + \boldsymbol{\omega} \times \boldsymbol{\omega}_{,\dot{q}} \\
= {} & \dot{\mathbf{b}}_2 \cdot \left(\mathbf{b}_{3,q}\mathbf{b}_1 + \mathbf{b}_3\mathbf{b}_{1,q}\right) + \dot{\mathbf{b}}_3 \cdot \left(\mathbf{b}_{1,q}\mathbf{b}_2 + \mathbf{b}_1\mathbf{b}_{2,q}\right) \\
& + \dot{\mathbf{b}}_1 \cdot \left(\mathbf{b}_{2,q}\mathbf{b}_3 + \mathbf{b}_2\mathbf{b}_{3,q}\right) + \dot{\mathbf{b}}_{2,q} \cdot \mathbf{b}_3\mathbf{b}_1 + \dot{\mathbf{b}}_{3,q} \cdot \mathbf{b}_1\mathbf{b}_2 \\
& + \dot{\mathbf{b}}_{1,q} \cdot \mathbf{b}_2\mathbf{b}_3 - \frac{d}{dt}(\mathbf{b}_{2,q}) \cdot \mathbf{b}_3\mathbf{b}_1 - \frac{d}{dt}(\mathbf{b}_{3,q}) \cdot \mathbf{b}_1\mathbf{b}_2 \\
& - \frac{d}{dt}(\mathbf{b}_{1,q}) \cdot \mathbf{b}_2\mathbf{b}_3 - \dot{\mathbf{b}}_{2,\dot{q}} \cdot \left(\dot{\mathbf{b}}_3\mathbf{b}_1 + \mathbf{b}_3\dot{\mathbf{b}}_1\right) \\
& - \dot{\mathbf{b}}_{3,\dot{q}} \cdot \left(\dot{\mathbf{b}}_1\mathbf{b}_2 + \mathbf{b}_1\dot{\mathbf{b}}_2\right) - \dot{\mathbf{b}}_{1,\dot{q}} \cdot \left(\dot{\mathbf{b}}_2\mathbf{b}_3 + \mathbf{b}_2\dot{\mathbf{b}}_3\right) + \boldsymbol{\omega} \times \boldsymbol{\omega}_{,\dot{q}} \\
= {} & \left\{\left(\dot{\mathbf{b}}_2 \cdot \dot{\mathbf{b}}_{3,\dot{q}} + \dot{\mathbf{b}}_{2,q} \cdot \mathbf{b}_3 - \dot{\mathbf{b}}_{2,q} \cdot \mathbf{b}_3 - \dot{\mathbf{b}}_{2,\dot{q}} \cdot \dot{\mathbf{b}}_3\right)\mathbf{b}_1\right. \\
& + \left(\dot{\mathbf{b}}_3 \cdot \dot{\mathbf{b}}_{1,\dot{q}} + \dot{\mathbf{b}}_{3,q} \cdot \mathbf{b}_1 - \dot{\mathbf{b}}_{3,q} \cdot \mathbf{b}_1 - \dot{\mathbf{b}}_{3,\dot{q}} \cdot \dot{\mathbf{b}}_1\right)\mathbf{b}_2 \\
& \left. + \left(\dot{\mathbf{b}}_1 \cdot \dot{\mathbf{b}}_{2,\dot{q}} + \dot{\mathbf{b}}_{1,q} \cdot \mathbf{b}_2 - \dot{\mathbf{b}}_{1,q} \cdot \mathbf{b}_2 - \dot{\mathbf{b}}_{1,\dot{q}} \cdot \dot{\mathbf{b}}_2\right)\mathbf{b}_3\right\} \\
& + \left\{\dot{\mathbf{b}}_2 \cdot \mathbf{b}_3\dot{\mathbf{b}}_{1,\dot{q}} + \dot{\mathbf{b}}_3 \cdot \mathbf{b}_1\dot{\mathbf{b}}_{2,\dot{q}} + \dot{\mathbf{b}}_1 \cdot \mathbf{b}_2\dot{\mathbf{b}}_{3,\dot{q}}\right\} \\
& - \left\{\dot{\mathbf{b}}_{2,\dot{q}} \cdot \mathbf{b}_3\dot{\mathbf{b}}_1 + \dot{\mathbf{b}}_{3,\dot{q}} \cdot \mathbf{b}_1\dot{\mathbf{b}}_2 + \dot{\mathbf{b}}_{1,\dot{q}} \cdot \mathbf{b}_2\dot{\mathbf{b}}_3\right\} + \boldsymbol{\omega} \times \boldsymbol{\omega}_{,\dot{q}} \\
= {} & \{[(\boldsymbol{\omega} \times \mathbf{b}_2) \cdot (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_3) - (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_2) \cdot (\boldsymbol{\omega} \times \mathbf{b}_3)]\,\mathbf{b}_1 \\
& + [(\boldsymbol{\omega} \times \mathbf{b}_3) \cdot (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_1) - (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_3) \cdot (\boldsymbol{\omega} \times \mathbf{b}_1)]\,\mathbf{b}_2 \\
& + [\boldsymbol{\omega} \times \mathbf{b}_1) \cdot (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_2) - (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_1) \cdot (\boldsymbol{\omega} \times \mathbf{b}_2)]\,\mathbf{b}_3\} \\
& + \{(\boldsymbol{\omega} \times \mathbf{b}_2) \cdot \mathbf{b}_3\,(\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_1) + (\boldsymbol{\omega} \times \mathbf{b}_3) \cdot \mathbf{b}_1\,(\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_2) \\
& + (\boldsymbol{\omega} \times \mathbf{b}_1) \cdot \mathbf{b}_2\,(\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_3)\} \\
& - \{(\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_2) \cdot \mathbf{b}_3\,(\boldsymbol{\omega} \times \mathbf{b}_1) + (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_3) \cdot \mathbf{b}_1\,(\boldsymbol{\omega} \times \mathbf{b}_2) \\
& + (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_1) \cdot \mathbf{b}_2\,(\boldsymbol{\omega} \times \mathbf{b}_3)\} + \boldsymbol{\omega} \times \boldsymbol{\omega}_{,\dot{q}} \\
= {} & \{[\boldsymbol{\omega} \cdot \mathbf{b}_2 \times (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_3) - \boldsymbol{\omega} \cdot \mathbf{b}_3 \times (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_2)]\,\mathbf{b}_1 \\
& + [\boldsymbol{\omega} \cdot \mathbf{b}_3 \times (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_1) - \boldsymbol{\omega} \cdot \mathbf{b}_1 \times (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_3)]\,\mathbf{b}_2 \\
& + [\boldsymbol{\omega} \cdot \mathbf{b}_1 \times (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_2) - \boldsymbol{\omega} \cdot \mathbf{b}_2 \times (\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_1)]\,\mathbf{b}_3\} \\
& + \{\boldsymbol{\omega} \cdot \mathbf{b}_2 \times \mathbf{b}_3\,(\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_1) + \boldsymbol{\omega} \cdot \mathbf{b}_3 \times \mathbf{b}_1\,(\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_2) \\
& + \boldsymbol{\omega} \cdot \mathbf{b}_1 \times \mathbf{b}_2\,(\boldsymbol{\omega}_{,\dot{q}} \times \mathbf{b}_3)\} \\
& - \{\boldsymbol{\omega}_{,\dot{q}} \cdot \mathbf{b}_2 \times \mathbf{b}_3\,(\boldsymbol{\omega} \times \mathbf{b}_1) + \boldsymbol{\omega}_{,\dot{q}} \cdot \mathbf{b}_3 \times \mathbf{b}_1\,(\boldsymbol{\omega} \times \mathbf{b}_2) \\
& + \boldsymbol{\omega}_{,\dot{q}} \cdot \mathbf{b}_1 \times \mathbf{b}_2\,(\boldsymbol{\omega} \times \mathbf{b}_3)\} + \boldsymbol{\omega} \times \boldsymbol{\omega}_{,\dot{q}} \\
= {} & \{[-\boldsymbol{\omega} \cdot (\mathbf{b}_2 \cdot \boldsymbol{\omega}_{,\dot{q}})\,\mathbf{b}_3 + \boldsymbol{\omega} \cdot (\mathbf{b}_3 \cdot \boldsymbol{\omega}_{,\dot{q}})\,\mathbf{b}_2]\,\mathbf{b}_1 \\
& + [-\boldsymbol{\omega} \cdot (\mathbf{b}_3 \cdot \boldsymbol{\omega}_{,\dot{q}})\,\mathbf{b}_1 + \boldsymbol{\omega} \cdot (\mathbf{b}_1 \cdot \boldsymbol{\omega}_{,\dot{q}})\,\mathbf{b}_3]\,\mathbf{b}_2 \\
& + [-\boldsymbol{\omega} \cdot (\mathbf{b}_1 \cdot \boldsymbol{\omega}_{,\dot{q}})\,\mathbf{b}_2 + \boldsymbol{\omega} \cdot (\mathbf{b}_2 \cdot \boldsymbol{\omega}_{,\dot{q}})\,\mathbf{b}_1]\,\mathbf{b}_3\}
\end{aligned}
$$

$$+\{-\boldsymbol{\omega}\cdot\mathbf{b}_1\mathbf{b}_1\times\boldsymbol{\omega}_{,\dot{q}}-\boldsymbol{\omega}\cdot\mathbf{b}_2\mathbf{b}_2\times\boldsymbol{\omega}_{,\dot{q}}-\boldsymbol{\omega}\cdot\mathbf{b}_3\mathbf{b}_3\times\boldsymbol{\omega}_{,\dot{q}}\}$$
$$-\{-\boldsymbol{\omega}_{,\dot{q}}\cdot\mathbf{b}_1\mathbf{b}_1\times\boldsymbol{\omega}-\boldsymbol{\omega}_{,\dot{q}}\cdot\mathbf{b}_2\mathbf{b}_2\times\boldsymbol{\omega}-\boldsymbol{\omega}_{,\dot{q}}\cdot\mathbf{b}_3\mathbf{b}_3$$
$$\times\boldsymbol{\omega}\}+\boldsymbol{\omega}\times\boldsymbol{\omega}_{,\dot{q}}$$
$$=\{\boldsymbol{\omega}\times\boldsymbol{\omega}_{,\dot{q}}\}-\{\boldsymbol{\omega}\times\boldsymbol{\omega}_{,\dot{q}}\}+\boldsymbol{\omega}_{,\dot{q}}\times\boldsymbol{\omega}+\boldsymbol{\omega}\times\boldsymbol{\omega}_{,\dot{q}}=0$$

proving Eq. (99), and finally establishing that Eq. (87) is precisely the same as Lagrange's equations in the form found in Eq. (70).

*d. Relationship between Lagrange's equations and Newton-Euler equations for rigid bodies.* Now we are well prepared to make comparisons of Lagrange's equations and the various Newton-Euler formulations that appear in the modern literature (in particular see Refs. 12 to 15, 18, 26, and 28 to 31, all of which deal with multiple-rigid body idealizations). The Newton-Euler formulations of generic equations used so widely today all involve linear combinations of dot-products of the basic vector equations of translation and rotation. While it may seem impossible to establish a parallel between Lagrange's equations and some of the ingeniously manipulated vector equations found in the modern literature, it is for the multiple-rigid body idealizations often a simpler matter to compare these manipulated vector equations to Eq. (87), which we have shown to be identical to Lagrange's equations even if very different in appearance. This observation will greatly facilitate the process of comparison in Section IV of this report.

*e. Structure of the matrix differential equations of motion.* For purposes of numerical integration, it is often convenient to rewrite the equations of motion (Eq. 70) in state variable form as a first order matrix differential equation. To this end, we define the $2n$ by 1 matrix

$$x \triangleq \begin{bmatrix} q \\ \hline \dot{q} \end{bmatrix} \tag{103}$$

and write Eq. (70) in the form

$$P\dot{x} = F \tag{104}$$

where $P$ and $F$ may depend on $x$ and $t$. Any set of second order differential equations can be written in the form of Eq. (104), but when the starting point is Eq. (70) we can guarantee certain properties of the matrix $P$ that facilitate numerical integration.

Equation (70) requires as input only $T = T(\dot{q}, q, t)$ and $Q_k (k = 1, \cdots, n)$. If we begin with $T$ in the form of Eq. (71) and replace $\mathbf{R}$ by the expansion found in Eq. (18), we have

$$\begin{aligned} T &= \frac{1}{2}\int\left[\sum_{k=1}^{n}\frac{\partial\mathbf{R}}{\partial q_k}\dot{q}_k+\frac{\partial\mathbf{R}}{\partial t}\right]\cdot\left[\sum_{k=1}^{n}\frac{\partial\mathbf{R}}{\partial q_k}\dot{q}_k+\frac{\partial\mathbf{R}}{\partial t}\right]dm \\ &= \frac{1}{2}\int\sum_{j=1}^{n}\frac{\partial\mathbf{R}}{\partial q_j}\cdot\sum_{k=1}^{n}\frac{\partial\mathbf{R}}{\partial q_k}\,dm\,\dot{q}_j\dot{q}_k+\int\sum_{j=1}^{n}\frac{\partial\mathbf{R}}{\partial q_j}\cdot\frac{\partial\mathbf{R}}{\partial t}\,dm\,\dot{q}_j+\frac{1}{2}\int\frac{\partial\mathbf{R}}{\partial t}\cdot\frac{\partial\mathbf{R}}{\partial t}\,dm \\ &\triangleq T_2+T_1+T_0 \end{aligned} \tag{105}$$

where $T_i$ consists only of terms in $i$th degree in the generalized velocities $\dot{q}_1, \cdots, \dot{q}_n$. The scalars $T_2$, $T_1$, and $T_0$ may be written in matrix form (using the notation of

Appendix A) as

$$T_2 = \frac{1}{2} \dot{q}^T M \dot{q} \tag{106a}$$

with

$$M \triangleq \int R^T_{,q} R_{,q^T}\, dm$$

an $n$ by $n$ symmetric matrix,

$$T_1 = \dot{q}^T \Gamma = \Gamma^T \dot{q} \tag{106b}$$

with

$$\Gamma \triangleq \int R^T_{,q} R_{,t}\, dm$$

an $n$ by 1 matrix, and

$$T_0 = \frac{1}{2} \int R^T_{,t} R_{,t}\, dm \tag{106c}$$

a scalar. Note that $M$, $\Gamma$, and $T_0$ can depend on $q$ and $t$, but not on $\dot{q}$ or any higher time derivatives of $q$.

Substituting $T$ into Eq. (70b) provides

$$\frac{d}{dt}(M\dot{q} + \Gamma) - \frac{\partial}{\partial q}\left(\frac{1}{2}\dot{q}^T M \dot{q} + \Gamma^T \dot{q} + T_0\right) = Q$$

or

$$M\ddot{q} + \dot{M}\dot{q} + \dot{\Gamma} - \frac{1}{2}(\dot{q}^T M)_{,q}\dot{q} - \Gamma^T_{,q}\dot{q} - T_{0,q} = Q \tag{107}$$

Substitute into Eq. (107) the expansions

$$\dot{\Gamma} = \Gamma_{,q^T}\dot{q} + \Gamma_{,t}$$

and (with the summation convention)

$$\begin{aligned} \dot{M}\dot{q} &= \{M_{\alpha\beta,q_\gamma}\dot{q}_\gamma \dot{q}_\beta\} + M_{,t}\dot{q} \\ &= \{\dot{q}^T (M\dot{q})_{\alpha,q}\} + M_{,t}\dot{q} \end{aligned}$$

where $(M\dot{q})_\alpha$ is the scalar in row $\alpha$ of $M\dot{q}$, and $\dot{q}^T(M\dot{q})_{\alpha,q}$ defines the scalar in row $\alpha$ of the $n$ by 1 matrix $\{\dot{q}^T (M\dot{q})_{\alpha,q}\}$. This convention permits the substitution

$$-\frac{1}{2}(\dot{q}^T M)_{,q}\dot{q} = -\frac{1}{2}\{\dot{q}^T M_{,q_\alpha}\dot{q}\}$$

and Eq. (107) becomes

$$M\ddot{q} + M_{,t}\dot{q} + \Gamma_{,t} + (\Gamma_{,q^T} - \Gamma^T_{,q})\dot{q} + \{\dot{q}^T(M\dot{q})_{\alpha,q}\} - \frac{1}{2}\{\dot{q}^T M_{,q_\alpha}\dot{q}\} = Q + T_{0,q} \tag{108}$$

The quantities in braces combine in an interesting way when written in terms of the summation convention and interpreted in light of the definition of $M$ in Eq. (106). Thus

$$\{\dot{q}^T (M\dot{q})_{\alpha,q}\} - \frac{1}{2}\{\dot{q}^T M_{,q_\alpha}\dot{q}\} = \{\dot{q}_\beta (M_{\alpha\gamma}\dot{q}_\gamma)_{,q_\beta}\} - \frac{1}{2}\{\dot{q}_\beta M_{\beta\gamma,q_\alpha}\dot{q}_\gamma\}$$
$$= \left\{M_{\alpha\gamma,q_\beta}\dot{q}_\beta\dot{q}_\gamma - \frac{1}{2} M_{\beta\gamma,q_\alpha}\dot{q}_\beta\dot{q}_\gamma\right\}$$
$$= \left\{\left(M_{\alpha\gamma,q_\beta} - \frac{1}{2} M_{\beta\gamma,q_\alpha}\right)\dot{q}_\beta\dot{q}_\gamma\right\}$$

and from Eq. (106)

$$\left(M_{\alpha\gamma,q_\beta} - \frac{1}{2} M_{\beta\gamma,q_\alpha}\right)\dot{q}_\beta\dot{q}_\gamma = \int\left(\frac{\partial^2 R^T}{\partial q_\alpha \partial q_\beta}\frac{\partial R}{\partial q_\gamma} + \frac{\partial R^T}{\partial q_\alpha}\frac{\partial^2 R}{\partial q_\gamma \partial q_\beta}\right.$$
$$\left. - \frac{1}{2}\frac{\partial^2 R^T}{\partial q_\beta \partial q_\alpha}\frac{\partial R}{\partial q_\gamma} - \frac{1}{2}\frac{\partial R^T}{\partial q_\beta}\frac{\partial^2 R}{\partial q_\gamma \partial q_\alpha}\right)\dot{q}_\beta\dot{q}_\gamma\, dm$$
$$= \int\left(\frac{1}{2}\frac{\partial^2 R^T}{\partial q_\alpha \partial q_\beta}\frac{\partial R}{\partial q_\gamma} - \frac{1}{2}\frac{\partial^2 R^T}{\partial q_\gamma \partial q_\alpha}\frac{\partial R}{\partial q_\beta}\right.$$
$$\left. + \frac{\partial R^T}{\partial q_\alpha}\frac{\partial^2 R}{\partial q_\gamma \partial q_\beta}\right)\dot{q}_\beta\dot{q}_\gamma\, dm$$
$$= \int\frac{\partial R^T}{\partial q_\alpha}\frac{\partial^2 R}{\partial q_\gamma \partial q_\beta}\, dm\, \dot{q}_\beta\dot{q}_\gamma \triangleq \dot{q}^T J^\alpha \dot{q}$$

where the $n$ by $n$ matrix

$$[J^\alpha_{\gamma\beta}] \triangleq \int\frac{\partial R^T}{\partial q_\alpha}\frac{\partial^2 R}{\partial q_\gamma \partial q_\beta}\, dm$$

joins $M$ and $\Gamma$ as a fundamental characteristic of the system.

Lagrange's equation then becomes

$$M\ddot{q} + M_{,t}\dot{q} + \Gamma_{,t} + G\dot{q} + \{\dot{q}^T J^\alpha \dot{q}\} = Q + T_{0,q} \tag{109}$$

with the definition of the skew-symmetric matrix

$$G \triangleq \Gamma_{,q^T} - \Gamma^T_{,q}$$

To obtain Lagrange's equations in the state equation form established by Eq. (104), we can define $u \triangleq \dot{q}$ and write

$$\left[\begin{array}{c|c} U & 0 \\ \hline 0 & M \end{array}\right]\left\{\begin{array}{c} \dot{q} \\ \hline \dot{u} \end{array}\right\} = \left\{\begin{array}{c} u \\ \hline Q + T_{0,q} - M_{,t}u - \Gamma_{,t} - Gu - \{u^T J^\alpha u\} \end{array}\right\} \tag{110}$$

where $U$ is the unit matrix. Thus the matrix $P$ in Eq. (104) is symmetric and block diagonal with only one $n$ by $n$ fully populated block $M$, which may depend upon $q$ and $t$.

Interest often focuses on the special case in which $\mathcal{L}_{,t} = 0$ and the equations of motion are linearized about the null solution $q \equiv 0$ (so $F(x, t) = 0$ for $x = 0$ in Eq. (104)). In seeking the linearized counterpart to the second order Eqs. (107), we can obviously ignore the wholly nonlinear terms $\dot{M}\dot{q}$ and $\frac{1}{2}(\dot{q}^T M)_{,q}\dot{q}$, but the remaining terms generally have both linear and nonlinear parts. To accomplish the necessary separation we can expand the functions $M$, $\Gamma$, and $T_0$ in Taylor series in $q$ about $q = 0$, truncating each series as required for linearization of Eq. (107). If we use a superscript to identify the degree of the $q$-terms in the expressions in the series, so that we have

$$M = M^0 + M^1 + M^2 + \cdots \tag{111a}$$

$$\Gamma = \Gamma^0 + \Gamma^1 + \Gamma^2 + \cdots \tag{111b}$$

$$T_0 = T_0^0 + T_0^1 + T_0^2 + \cdots \tag{111c}$$

then inspection of Eq. (107) indicates that linearization requires only

$$M \cong M^0 \tag{112a}$$

$$\Gamma \cong \Gamma^0 + \Gamma^1 \tag{112b}$$

$$T_0 \cong T_0^0 + T_0^1 + T_0^2 \tag{112c}$$

Since $\Gamma^1$ is an $n$ by 1 matrix, and $T_0^2$ is a scalar, we can adopt the expansions

$$\Gamma^1 \triangleq gq \tag{113}$$

$$T_0^2 \triangleq \frac{1}{2} q^T \kappa q \tag{114}$$

with $g$ and $\kappa$ constant $n$ by $n$ matrices, and with $\kappa$ symmetric.

Substitution of these approximations into Eq. (109) leaves the intermediate result

$$M^0\ddot{q} + g\dot{q} - g^T\dot{q} - \kappa q = Q + T^1_{0,q} \tag{115}$$

in which the identities $T^0_{0,q} = 0$, $\dot{\Gamma}^0 = 0$, and $\Gamma^{0T}_{,q} = 0$ have been noted. To get this equation in its final linearized form, we must introduce an approximation of $Q$, which has heretofore been unrestricted. As long as we restrict attention to the case in which $q \equiv 0$ is a solution to the equations of motion (as is required if the linearized variational equation is to have any formal significance), then any terms in $Q$ that are independent of $q$ and $\dot{q}$ must cancel the term $T^1_{0,q}$ in Eq. (115). As long as $Q$ contains no sublinear terms in $q$ or $\dot{q}$ (involving powers less than one), we can replace the right side of Eq. (115) by $-kq - D\dot{q}$. Then with the definition of the skew symmetric $n$ by $n$ matrix

$$G^0 \triangleq g - g^T \tag{116a}$$

and the $n$ by $n$ matrix

$$K \triangleq -\kappa + k \tag{116b}$$

we can write the linearized variational equation as

$$M^0\ddot{q} + G^0\dot{q} + D\dot{q} + Kq = 0 \tag{117}$$

In the special case for which the generalized force $Q$ is a combination of a *conservative force* $Q^c = -V_{,q}$ for some scalar potential energy $V$, the matrix $k$ (and hence the matrix $K$) is symmetric. If instead $Q$ is a combination of a conservative force $Q^c$ plus a restricted *Rayleigh damping force* $Q^d = -\mathcal{R}_{,\dot{q}}$ where $\mathcal{R} = \frac{1}{2}\dot{q}^T D\dot{q}$, then $D$ is taken as a symmetric damping matrix. In general, however, neither $D$ nor $K$ need be symmetric.

In state-variable form, Eq. (117) may be written

$$\left[\begin{array}{c|c} U & 0 \\ \hline 0 & M^0 \end{array}\right] \begin{Bmatrix} \dot{q} \\ \dot{u} \end{Bmatrix} = \left[\begin{array}{c|c} 0 & U \\ \hline -K & -G^0 - D \end{array}\right] \begin{Bmatrix} q \\ u \end{Bmatrix} \tag{118a}$$

where $u \triangleq \dot{q}$.

The alternative form

$$\left[\begin{array}{c|c} K & 0 \\ \hline 0 & M^0 \end{array}\right] \begin{Bmatrix} \dot{q} \\ \dot{u} \end{Bmatrix} = \left[\begin{array}{c|c} 0 & K \\ \hline -K & -G^0 - D \end{array}\right] \begin{Bmatrix} q \\ u \end{Bmatrix} \tag{118b}$$

may prove advantageous when $K$ is symmetric and $D \equiv 0$, since then the constant coefficient matrices on the left and right hand sides are respectively symmetric and skew symmetric. (Advantages of this structure are noted in Ref. 31, pp. 50–55.)

**2. Lagrange's equations for simply constrained systems.** Lagrange's equations in the form of Eqs. (70), (109), and (110) are directly applicable only if there exists a set of independent generalized coordinates. If redundant variables are employed in the presence of this condition, one can retain the form of Eq. (70d) as shown in Ref. 43. We recognize however that in many problems the kinematical variables that most naturally emerge in the description of the motion of a system are not independent, being interrelated by some kind of constraint equation. We have noted that if the generalized coordinates $q_1, \cdots, q_\nu$ are related by $m$ *holonomic* constraints (in the form of Eq. (9)), then it is always possible (if sometimes difficult) to solve the constraint equations for $m$ of the variables in terms of those remaining, in order to find $\nu - m$ independent generalized coordinates. Then we can let $\nu - m$ be $n$ and use Eq. (70). We have also noted, however, that this process of algebraic reduction from $\nu$ interdependent variables to $n$ independent variables is not possible for nonholonomic systems, so for systems in this class Lagrange's equations in the forms presented in the preceding section must be abandoned entirely.

In the restricted (but commonplace) case in which the constraints among the $\nu$ generalized coordinates can be written in the simple form (as in Subsection II-A-3)

$$(54) \qquad \sum_{k=1}^{\nu} A_{sk}\dot{q}_k + B_s = 0 \qquad s = 1, \cdots, m$$

where $A_{sk} = A_{sk}(q_1, \cdots, q_\nu, t)$ and $B_s = B_s(q_1, \cdots, q_\nu, t)$, we have established in Subsections II-A-3 and II-A-4 that one can succeed in developing variations of

D'Alembert's principle that produce $\nu$ equations for the determination of the time history of the $\nu$ generalized coordinates. (Constraint equations in the form of Eq. (54) are called *simple* or *Pfaffian.*) In the present section we will explore the possibility of generalizing Eq. (70) to accomplish the same objective.

Lagrange's equation in the form of Eq. (70) came originally from Eq. (8), repeated here as

$$\sum_{k=1}^{\nu}\sum_{j=1}^{N}(\mathbf{F}_j - m_j\ddot{\mathbf{R}}_j)\cdot\frac{\partial \mathbf{R}_j}{\partial q_k}\,\delta q_k = 0 \tag{119}$$

In the derivation of Eq. (70) we took advantage of the fact that we could treat each of the generalized virtual displacements independently without violating any constraints, and set the coefficient of $\delta q_k$ in Eq. (119) equal to zero for each value of $k$; in the process we discovered that certain unknown and unwanted *non-working constraint forces* disappeared from the problem.

Because the generalized virtual displacements $\delta q_1, \cdots, \delta q_\nu$ are simply imaginary quantities, we are still free to conceive them as independent quantities, and from Eq. (119) write

$$\sum_{j=1}^{N}(\mathbf{F}_j - m_j\ddot{\mathbf{R}}_j)\cdot\frac{\partial \mathbf{R}_j}{\partial q_k} = 0 \qquad k = 1,\cdots,\nu \tag{120}$$

These $\nu$ independent generalized virtual displacements imposed on the system are not compatible with constraints, however, and as a consequence the constraint forces that disappeared from the equations in the case of independent generalized coordinates will no longer disappear. We can still separate $\mathbf{F}_j$ into the two parts $\mathbf{f}_j$ and $\mathbf{f}'_j$, where the latter represent forces of constraint, and write Eq. (120) as

$$\sum_{j=1}^{N}\mathbf{f}_j\cdot\frac{\partial \mathbf{R}_j}{\partial q_k} + \sum_{j=1}^{N}\mathbf{f}'_j\cdot\frac{\partial \mathbf{R}_j}{\partial q_k} = \sum_{j=1}^{N} m_j\ddot{\mathbf{R}}_j\cdot\frac{\partial \mathbf{R}_j}{\partial q_k} \qquad k = 1,\cdots,\nu \tag{121}$$

but each of the three sums in Eq. (121) is generally nonzero. For contrast, compare Eqs. (11), (12), and (13) to Eqs. (120) and (121). It is customary to designate the two sums on the left side of Eq. (121) as the $k$th *generalized active force* $Q_k$ and the $k$th *generalized constraint force* $Q'_k$ respectively. We can use Eq. (69) to replace the right side of Eq. (121), and write

$$Q_k + Q'_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} \qquad k = 1,\cdots,\nu \tag{122}$$

The liability of Eq. (122) lies in the fact that $Q'_k$ are generally unknown. To eliminate these terms from the equations we must use the constraint equations in Eq. (54).

In differential form, Eq. (54) becomes

$$\sum_{k=1}^{\nu} A_{sk}\,dq_k + B_s\,dt = 0 \qquad s = 1,\cdots,m \tag{123}$$

If we wish to impose on the generalized coordinates a set of virtual displacements that are compatible with constraints, we must impose the constraint

$$\sum_{k=1}^{\nu} A_{sk}\,\delta q_k = 0 \qquad s = 1, \cdots, m \tag{124}$$

noting that there is no variation in $t$ during an imaginary or virtual displacement. Had we imposed this constraint upon the quantities $\delta q_k$ in Eq. (119), we would have no work done by the constraint forces, so that

$$\sum_{k=1}^{\nu}\sum_{j=1}^{N} \mathbf{f}'_j \cdot \frac{\partial \mathbf{R}_j}{\partial q_k}\delta q_k = 0$$

or

$$\sum_{k=1}^{\nu} Q'_k \delta q_k = 0 \tag{125}$$

**This step does not permit the conclusion that $Q'_k = 0$ for $k = 1, \cdots, \nu$, because the quantities $\delta q_k (k = 1, \cdots, \nu)$ in Eq. (125) are related by Eq. (124). We might choose $\nu - m$ of these quantities arbitrarily, say $\delta q_{m+1}, \cdots, \delta q_\nu$, but the remaining $m$ quantities are determined by Eq. (124). Our next objective is to use Eqs. (124) and (125) together to obtain a solution for $Q'_k$, which we can substitute into the equations of motion (Eq. 122) in order to eliminate this unknown generalized constraint force from the equations. To this end, we introduce $m$ new unknowns $\lambda_1, \cdots, \lambda_m$ (called *Lagrange multipliers*); we multiply $\lambda_s$ by Eq. (124) for $s = 1, \cdots, m$, and add the results to obtain (after reversing the summation sequence)**

$$\sum_{k=1}^{\nu}\sum_{s=1}^{m} \lambda_s A_{sk}\,\delta q_k = 0 \tag{126}$$

Clearly this equation is valid no matter what values we attach to $\lambda_1, \cdots, \lambda_m$, and since this double sum is zero there can be no harm in adding it to Eq. (125) to obtain

$$\sum_{k=1}^{\nu}\left[Q'_k + \sum_{s=1}^{m} \lambda_s A_{sk}\right]\delta q_k = 0 \tag{127}$$

We are now free to choose $\lambda_1, \cdots, \lambda_m$ such that the coefficients of $\delta q_1, \cdots, \delta q_m$ are zero, or

$$Q'_k + \sum_{s=1}^{m} \lambda_s A_{sk} = 0 \qquad k = 1, \cdots, m \tag{128a}$$

Then we are left in Eq. (127) with only terms involving the $\nu - m$ quantities $\delta q_{m+1}, \cdots, \delta q_\nu$, and these we are free to treat as independent variables. Thus Eq. (127) becomes

$$\sum_{k=m+1}^{\nu}\left[Q'_k + \sum_{s=1}^{m} \lambda_s A_{sk}\right]\delta q_k = 0$$

implying that

$$Q'_k + \sum_{s=1}^{m} \lambda_s A_{sk} = 0 \qquad k = m+1, \cdots, \nu \tag{128b}$$

Eqs. (128a, b) provide the desired solution

$$Q'_k = -\sum_{s=1}^{m} \lambda_s A_{sk} \qquad k = 1, \cdots, \nu \tag{129}$$

Substituting this result into Eq. (122) provides the required equations of motion in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k - \sum_{s=1}^{m} \lambda_s A_{sk} \qquad k = 1, \cdots, \nu \tag{130}$$

Because the Lagrange multipliers $\lambda_1, \cdots, \lambda_m$ introduce $m$ unknowns into the problem in addition to the $\nu$ generalized coordinates, Eq. (130) does not in itself constitute a complete set of governing equations. These equations must be solved in conjunction with the kinematical constraint equations as presented in Eq. (54). In combining these equations, it is most convenient to work with the constraint equations in their matrix form (as in Eq. (55))

$$A\dot{q} + B = 0 \tag{55}$$

and to recast the dynamical equations (Eq. (130)) in the matrix form

$$\frac{d}{dt}(T_{,\dot{q}}) - T_{,q} = Q - A^T \lambda \tag{131}$$

where $\lambda$ is the $m$ by 1 matrix with elements $\lambda_1, \cdots, \lambda_m$. After noting the equivalence between Eq. (70b) and Eq. (107), we can write Eq. (131) in the more explicit form

$$M\ddot{q} + A^T\lambda = Q + T_{0,q} - \dot{\Gamma} + \left[\left(\frac{1}{2}\dot{q}^T M\right)_{,q} - \dot{M} + \Gamma^T_{,q}\right]\dot{q} \tag{132}$$

and then combine Eqs. (132) and (55) in the system equation

$$\left[\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & M & A^T \\ \hline U & 0 & 0 \end{array}\right] \left\{\begin{array}{c} \dot{q} \\ \hline \dot{u} \\ \hline \lambda \end{array}\right\} = \left\{\begin{array}{c} -B \\ \hline Q + T_{0,q} - \dot{\Gamma} + \left[\left(\frac{1}{2}u^T M\right)_{,q} - \dot{M} + \Gamma^T_{,q}\right]u \\ \hline u \end{array}\right\} \tag{133}$$

where again $u \triangleq \dot{q}$, and the terms $\dot{M}$ and $\dot{\Gamma}$ can be expanded as after Eq. (107).

It is the $2\nu + m$ scalar equations in Eq. (133) that must be subjected to numerical integration. The coefficient matrix on the left side is not well designed to facilitate this operation.

Equation (133) applies to simple nonholonomic systems, so that among the methods developed in this report it finds its competition in Lagrange's form of D'Alembert's principle for simple nonholonomic systems (Eqs. (60) and (54) of Subsection A-3), and Kane's quasi-coordinate formulation of D'Alembert's principle (Eqs. (66b), (55), and (64c), collected at the end of Subsection A-4). Since Lagrange's method is a special case of Kane's, it will suffice to compare Eq. (133) to those collected at the end of Subsection A-4. The most obvious difference is in the number of equations. Whereas Kane requires only $2\nu - m$ scalar equations, the Lagrange multiplier method involves $2\nu + m$, sufficient to solve for $q_1, \cdots, q_\nu$, and $\dot{q}_1, \cdots, \dot{q}_\nu$ and $\lambda_1, \cdots, \lambda_m$. Unless all of these variables are required as problem outputs, Kane's method has an obvious advantage for constrained systems. In application to unconstrained systems Kane's approach still has attractive features for some problems because of the possibility of choosing the $u$'s as the kinematic variables, but in this case it must compete with Lagrange's generalized coordinate equations in the form of Eq. (110). The explicit structure of Eqs. (110) and (133) contrasts with the collection of equations (66b), (55), and (64c) that represent Kane's method. Because the expressions for $f$ and $f^*$ depend upon the physical system under consideration, it is somewhat more difficult to cast Kane's equations once and for all in a generic form. This flexibility in formulation can be an important advantage to the analyst seeking an efficient ad hoc approach to a given problem, but the formal explicit structure of the Lagrangian generalized coordinate equations may offer advantages when the objective is a multipurpose computer program.

**3. Lagrange's quasi-coordinate equations.** In the two subsections preceding, Lagrange's equations have been written in terms of a system of generalized coordinates, which we have defined as a set of scalar functions of time $q_1, \cdots, q_\nu$ that fully define the configuration of the system at all points in time. In Subsection B-1 these scalars are independent, and in Subsection B-2 they are related by a set of constraint equations, but in either case the equations of motion emerge as second order differential equations in the $q$'s. Moreover, the basic differential equations in these two cases (typified by Eqs. (70) and (130)) involve in their final form only scalars, and specifically include partial derivatives of the kinetic energy $T$. (Of course one might use vector analysis to find the velocities leading to $T$ or to the generalized forces in these equations, but these vector operations are not explicit in the final equations.)

Prior to the development of Lagrange's equations, we examined in Section A various manifestations of D'Alembert's principle. In these equations, vector operations are much more explicit, although the final equations (as represented by Eqs. (20) and (60), for two examples) are scalar equations. In these equations the kinetic energy $T$ does not appear, and instead of taking partial derivatives of $T$ the analyst is obliged to find inertial acceleration vectors and certain of their dot products.

One of the seemingly apparent advantages of D'Alembert's principle is in the flexibility it offers in the selection of coordinates; there is no restriction to generalized coordinates, and in Subsection A-4 we noted that it is possible in this context to obtain equations of motion in terms of derivatives of *quasi-coordinates*, even for simple nonholonomic systems. These quantities were defined previously as linear combinations of $n$ selected generalized velocities in Eq. (64), but here we must broaden the definition to include all $\nu$ generalized velocities as follows:

$$u_k \triangleq \sum_{s=1}^{\nu} W_{sk}\dot{q}_s + w_k, \qquad k = 1, \cdots, \nu \tag{134a}$$

and in matrix form:

$$u = W^T\dot{q} + w \tag{134b}$$

and

$$u^T = \dot{q}^T W + w^T \tag{134c}$$

where now $u$, $\dot{q}$, and $w$ are $\nu$ by 1 matrices, and $W$ is a $\nu$ by $\nu$ matrix, which, by the conventions of Appendix A, may be written as

$$W \triangleq \frac{\partial u^T}{\partial \dot{q}} \triangleq u^T_{,\dot{q}} \tag{134d}$$

Recall that $n$ is the number of degrees of freedom for a system defined by $\nu$ generalized coordinates, and $m \triangleq \nu - n$ is the number of constraint equations of the simple (Pfaffian) form

$$\sum_{k=1}^{\nu} A_{sk}\dot{q}_k + B_s = 0, \qquad s = 1, \cdots, m \tag{135a}$$

or

$$A\dot{q} + B = 0 \tag{135b}$$

as originally presented in Eqs. (54) and (55). The scalars $W_{sk}$, $w_k$, $A_{sk}$, and $B_s$ (and hence the matrices $W$, $w$, $A$, and $B$) may depend upon $q_1, \cdots, q_\nu$ and $t$.

The objective in the present section is to demonstrate that Lagrange's equations can *also* be restated in terms of quasi-coordinate derivatives, thereby recapturing the equivalence of scope of the two basic methods identified in this report as generalizations of D'Alembert's principle and Lagrange's equations. To accomplish this objective, we can begin with Eq. (130), and use Eqs. (134) and (135) to manipulate the equations of motion so as to remove all time derivatives of the generalized coordinates, replacing them with quasi-coordinate derivatives. Special cases of the required manipulations can be found in Refs. 39, 44, and 45.

As in Subsection A-4, the matrix $W$ in Eq. (134b) is assumed to be nonsingular, permitting the solution of this equation for $\dot{q}$ to be written as

$$\dot{q} = (W^T)^{-1}(u - w) \tag{136}$$

Recall that $\dot{q}$ is a $\nu$ by 1 matrix; it is assumed here that in contrast with the procedure in Subsection A-4 the constraint equations (135) have *not* been employed to express the $m = \nu - n$ generalized velocities $\dot{q}_{n+1}, \cdots, \dot{q}_\nu$ in terms of the selected set $\dot{q}_1, \cdots, \dot{q}_n$.

Thus if we have the kinetic energy $T$ for Eq. (130) written in terms of the $\nu$ generalized coordinates in the $\nu$ by 1 matrix $q$, their first time derivatives, and time $t$, we can use Eq. (136) to obtain the kinetic energy in terms of $u$, $q$, and $t$; this expression we call $\overline{T}$ so that

$$T(\dot{q}, q, t) \triangleq \overline{T}(u, q, t) \tag{137}$$

When Eq. (130), or its matrix counterpart Eq. (131), requires $T_{,\dot{q}} \triangleq \partial T/\partial \dot{q}$, we can write

$$\frac{\partial T}{\partial \dot{q}} = \frac{\partial u^T}{\partial \dot{q}} \frac{\partial \overline{T}}{\partial u} \tag{138}$$

Since from Eq. (134b) we have

$$u^T = \dot{q}^T W + w^T \tag{139}$$

Eq. (131) requires

$$T_{,\dot{q}} \triangleq \frac{\partial T}{\partial \dot{q}} = W \frac{\partial \overline{T}}{\partial u} \triangleq W\overline{T}_{,u} \tag{140}$$

The term $T_{,q}$ in Eq. (131) similarly becomes

$$T_{,q} \triangleq \frac{\partial T}{\partial q} = \frac{\partial \overline{T}}{\partial q} + \frac{\partial u^T}{\partial q}\frac{\partial \overline{T}}{\partial u} \tag{141}$$

With Eq. (139), this term in Eq. (131) becomes

$$\begin{aligned} T_{,q} &= \frac{\partial \overline{T}}{\partial q} + \left[\frac{\partial}{\partial q}(\dot{q}^T W) + \frac{\partial w^T}{\partial q}\right]\frac{\partial \overline{T}}{\partial u} \\ &= \overline{T}_{,q} + [(\dot{q}^T W)_{,q} + w^T_{,q}]\,\overline{T}_{,u} \end{aligned} \tag{142}$$

Now we can rewrite Eq. (131) in the intermediate form

$$\frac{d}{dt}(W\overline{T}_{,u}) - \overline{T}_{,q} - [(\dot{q}^T W)_{,q} + w^T_{,q}]\,\overline{T}_{,u} = Q - A^T\lambda \tag{143}$$

Before Eq. (143) is ready for incorporation into a numerical integration program, the product differentiations must be separated and the explicit $\dot{q}$ then removed in favor of $u$. The first term, for example, can be written as

$$\frac{d}{dt}(W\overline{T}_{,u}) = \left(\frac{d}{dt}W\right)\overline{T}_{,u} + W\frac{d}{dt}(\overline{T}_{,u}) = \left[\frac{dW}{dt}\right]\overline{T}_{,u} + W\frac{d}{dt}(\overline{T}_{,u})$$

Since $W$ depends only on $q$ and possibly $t$, the time derivative of its element $W_{ij}$ can be expanded as

$$\frac{d}{dt}W_{ij} = \dot{q}^T\frac{\partial W_{ij}}{\partial q} + \frac{\partial W_{ij}}{\partial t} = \dot{q}^T W_{ij,q} + W_{ij,t}$$

permitting the observation

$$\begin{aligned} \frac{d}{dt}(W\overline{T}_{,u}) &= [\dot{q}^T W_{ij,q} + W_{ij,t}]\,\overline{T}_{,u} + W\frac{d}{dt}(\overline{T}_{,u}) \\ &= [\dot{q}^T W_{ij,q}]\,\overline{T}_{,u} + W_{,t}\overline{T}_{,u} + W\frac{d}{dt}(\overline{T}_{,u}) \end{aligned} \tag{144a}$$

Similarly from Eq. (143) we have the expanded term

$$(\dot{q}^T W)_{,q} \overline{T}_{,u} = \left\{ \frac{\partial (\dot{q}^T W)}{\partial q_k} \frac{\partial \overline{T}}{\partial u} \right\} = \left\{ \dot{q}^T \frac{\partial W}{\partial q_k} \frac{\partial \overline{T}}{\partial u} \right\} = \{ \dot{q}^T W_{,q_k} \overline{T}_{,u} \} \tag{144b}$$

After substitution of Eqs. (144), Eq. (143) takes the form

$$\begin{aligned} W \frac{d}{dt} (\overline{T}_{,u}) &+ [\dot{q}^T W_{ij,q}] \, \overline{T}_{,u} + W_{,t} \overline{T}_{,u} - \overline{T}_{,q} \\ &- \{ \dot{q}^T W_{,q_k} \overline{T}_{,u} \} - w^T_{,q} \, \overline{T}_{,u} = Q - A^T \lambda \end{aligned} \tag{145}$$

Now we can invert Eq. (139) to obtain

$$\dot{q}^T = (u^T - w^T) \, W^{-1} \tag{146}$$

for substitution and elimination of explicit $\dot{q}$ from the equations of motion. The final result is obtained by combining Eqs. (145) and (146) in the form

$$\begin{aligned} W \frac{d}{dt} (\overline{T}_{,u}) &+ [(u^T - w^T) \, W^{-1} W_{ij,q}] \, \overline{T}_{,u} + W_{,t} \overline{T}_{,u} - w^T_{,q} \overline{T}_{,u} \\ &- \{ (u^T - w^T) \, W^{-1} W_{,q_k} \overline{T}_{,u} \} - \overline{T}_{,q} = Q - A^T \lambda \end{aligned} \tag{147a}$$

The burdens of notational conventions are severe at this point, so some reminder of the rules of Appendix A seems appropriate. Recall that braces enclose column matrices whenever the indexed element appears explicitly; thus the term in braces with index $k$ in Eq. (147a) is an explicit expression for the $k$th element in that column matrix. Similarly, square brackets enclose rectangular matrices whenever the indexed element appears explicitly; the term in brackets with indices $ij$ in Eq. (147) is an explicit expression for the element in the $i$th row and $j$th column of that rectangular (here square) matrix.

With some sacrifice of detail, we can symbolize the mathematical structure of Eq. (147a) more simply with the alternative representation

$$\frac{d}{dt} (\overline{T}_{,u}) + W^{-1} \gamma \overline{T}_{,u} - W^{-1} \overline{T}_{,q} = W^{-1} Q - W^{-1} A^T \lambda \tag{147b}$$

Here the elements of the new matrix $\gamma$ can be calculated (with some labor) from Eq. (147a).

The equations of motion (147) combine with the constraint Eq. (135b) and the kinematic equation (134b) to provide a complete set of $2\nu + m$ scalar first order differential equations in the $2\nu + m$ unknowns $q_1, \cdots, q_\nu$, $u_1, \cdots, u_\nu$, and $\lambda_1, \cdots, \lambda_m$. In programming for digital computation we would follow the model established by Eq. (133), replacing $\ddot{q}$ by $\dot{u}$, and filling in the middle set of equations from Eq. (147).

In comparing Lagrange's quasi-coordinate equations with Kane's formulation, as represented by the collection in Eqs. (66b), (55), and (64c), it is important to note that Kane utilizes only $2\nu - m$ first order equations in the $2\nu - m$ unknowns $q_1, \cdots, q_\nu$, $u_1, \cdots, u_{\nu-m}$. This reduction has been accomplished by making explicit use of the constraint equations to eliminate $m$ of the $\dot{q}$ terms from the problem, and the labors of this reduction should not go unnoticed. Nonetheless it is highly probable that once the alternative sets of equations are formulated the computational advantage lies with Kane's equations. (Of course if constraint forces are required as part of the computational output, the advantage might well shift to the Lagrangian formulation, because constraint forces follow from the Lagrange multipliers through Eq. (129).)

In the special case of an unconstrained system with $n$ independent generalized coordinates and quasi-coordinates, we can abandon Eq. (135b) and remove the term $A^T\lambda$ from Eq. (147), so that the result compares much more closely to Kane's formulation, as reflected now in Eqs. (66b) and (64c). Meaningful comparison of these approaches will have to await comparison of specific cases in later sections of this report. As we shall see, Eqs. (147b) and (66b) often give identical results; Eqs. (134b) and (64c) are obviously identical.

A comparison of Lagrange's quasi-coordinate equations (Eq. (147)), with Lagrange's generalized coordinate equations (Eq. (130)), or its matrix counterpart, reduces to the observation that they are identical for the special choice of quasi-coordinates $u_i \triangleq \dot{q}_i$, $i = 1, \cdots, \nu$. This choice implies that $w = 0$ and $W = U$ in Eq. (134b), resulting in the collapse of Eq. (147a) to the matrix counterpart of Eq. (130). Thus the quasi-coordinate approach subsumes the generalized coordinate approach; any problem can be reduced by a quasi-coordinate formulation to the same equations resulting from a generalized coordinate formulation, but the converse is not true.

## C. Hamilton's Equations

**1. Hamilton's equations for simply constrained systems.** In previous sections we have noted the desirability of obtaining the differential equations that characterize a system in the form of first order matrix differential equations. The adoption of this standard form may facilitate numerical integrations, stability analysis, and even analytical solution. Although in some instances the equations of motion derived in the preceding sections first emerged as second order differential equations, we have noted that it is always a straightforward task to recast them as first order equations (see the transition from Eq. (117) to Eq. (118), for an example). In the present section we will consider an alternative equation formulation procedure that will take us directly to first order equations.

The starting point for the proposed derivation is the set of scalar equations in Eq. (130), or equivalently the matrix Eq. (131), which is Lagrange's generalized coordinate equation for simply constrained systems. Before we depart significantly from this equation, however, we will introduce a minor modification that is particularly attractive when the generalized forces in $Q$ fall in the restricted class called *conservative*. (In spacecraft applications it is seldom appropriate to consider all generalized forces to be conservative, although usually some forces fall in this category.)

When there exists a scalar $V = V(q)$ such that

$$Q = -\frac{\partial V}{\partial q} \tag{148}$$

then the force system is said to be conservative, as noted in advance of Eq. (70c). More generally, we can separate $Q$ into a conservative part and a nonconservative part $\bar{Q}$, and write

$$Q = \bar{Q} - \frac{\partial V}{\partial q} \tag{149}$$

In substituting Eq. (149) into Eq. (131), it is convenient to introduce the Lagrangian $\mathcal{L}$ such that

$$\mathcal{L} \triangleq T - V \tag{150}$$

Since $V_{,\dot{q}} = 0$, Eq. (131) becomes

$$\frac{d}{dt}(\mathcal{L}_{,\dot{q}}) - \mathcal{L}_{,q} = \bar{Q} - A^T\lambda \tag{151}$$

The quantity $\mathcal{L}_{,\dot{q}}$ is by virtue of the structure of $T$ (see Eq. (105)) a linear (but not necessarily homogeneous) form in $\dot{q}$; its time derivative provides all of the second derivatives of $\dot{q}$ that appear in Eq. (151). By introducing a new set of variables called *generalized momenta* defined by

$$p_k \triangleq \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \triangleq \mathcal{L}_{,\dot{q}_k} \tag{152}$$

and defining the corresponding matrix $p \triangleq \{p_1, \cdots, p_\nu\}^T$, we can make the substitution

$$\mathcal{L}_{,\dot{q}} = p \tag{153}$$

and give Eq. (151) the structure of a first order set of differential equations as follows:

$$\dot{p} = \mathcal{L}_{,q} + \bar{Q} - A^T\lambda \tag{154}$$

The combination of Eqs. (153), (154), and the constraint equations (135b) is a completely viable representation of the system behavior, and it offers certain computational advantages over what follows (see Ref. 35). However, with a little effort we can improve its analytical structure further. As it stands, we can see from Eqs. (105) through (107) that Eq. (153) could be written as

$$p = M\dot{q} + \Gamma \tag{155}$$

The relationship between the generalized momenta in $p$ and the variables appearing in the matrix $u$ in Eq. (134b) should be noted. Since $\mathcal{L}$ is a quadratic form in $\dot{q}$, Eq. (155) is a special case of Eq. (134b), and $p$ is a special case of $u$. It is a particularly attractive choice, as evidenced by the structure of the left side of Eq. (154).

We can improve the structure of Eq. (153) without jeopardizing the attractive structure of Eq. (154) by replacing the Lagrangian $\mathcal{L}$ $(q,\dot{q},t)$ by a new scalar

$\mathcal{H} = \mathcal{H}(q,p,t)$ called the *Hamiltonian* and defined by

$$\mathcal{H} \triangleq \frac{\partial \mathcal{L}}{\partial \dot{q}^T} \dot{q} - \mathcal{L} = p^T \dot{q} - \mathcal{L} \tag{156a}$$

As an alternative expression equivalent to Eq. (156a), we can adopt

$$\mathcal{H} = \mathcal{L}_2 - \mathcal{L}_0 \tag{156b}$$

where subscripts on $\mathcal{L}$ identify the degree of the homogeneous forms in $\dot{q}$ in the expression

$$\mathcal{L} \triangleq \mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0$$

In terms of the potential energy $V$ and the kinetic energy $T = T_2 + T_1 + T_0$ (see Eq. (105)), the Hamiltonian becomes

$$\mathcal{H} = T_2 + V - T_0 \tag{156c}$$

Whichever of the expressions (156a, b, or c) is adopted, one must use the inverse of Eq. (155) to remove $\dot{q}$ from $\mathcal{H}$ before proceeding to construct the equations of motion. Then we have $\mathcal{H} = \mathcal{H}(q,p,t)$, and it is this interpretation we now adopt for the left side of Eq. (156a). Explicit expressions for $\mathcal{H}(q,p,t)$ follow from the substitution of Eq. (106a) and the inverse of Eq. (155) into Eq. (156c); the result is

$$\mathcal{H} = \frac{1}{2}\dot{q}^T M \dot{q} + V - T_0 = \frac{1}{2}(p^T - \Gamma^T) M^{-1} (p - \Gamma) + V - T_0 \tag{156d}$$

Differentiation of Eq. (156a) produces

$$\frac{d\mathcal{H}}{dt} = \frac{d}{dt}[p^T \dot{q} - \mathcal{L}(q, \dot{q}, t)]$$

or

$$\begin{aligned} \dot{q}^T \frac{\partial \mathcal{H}}{\partial q} + \dot{p}^T \frac{\partial \mathcal{H}}{\partial p} + \frac{\partial \mathcal{H}}{\partial t} &= \dot{p}^T \dot{q} + p^T \ddot{q} - \dot{q}^T \frac{\partial \mathcal{L}}{\partial q} - \ddot{q}^T \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial t} \\ &= \dot{p}^T \dot{q} - \dot{q}^T \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial \mathcal{L}}{\partial t} \end{aligned} \tag{157}$$

the final step being justified by the identity

$$p^T \ddot{q} - \ddot{q}^T \mathcal{L}_{,\dot{q}} = \ddot{q}^T p - \ddot{q}^T p = 0$$

Equation (157) can be written as

$$\dot{q}^T (\mathcal{H}_{,q} + \mathcal{L}_{,q}) + \dot{p}^T (\mathcal{H}_{,p} - \dot{q}) + \left(\frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{L}}{\partial t}\right) = 0 \tag{158}$$

but the interdependence of the terms in this equation does not yet permit the conclusion that the three terms in parentheses are all individually zero.

If we consider virtual displacements, however, we need not acknowledge the presence of constraints or the passage of time, and we can write

$$\delta\mathcal{H}(q,p,t) = \delta\,[p^T\dot{q} - \mathcal{L}(q,\dot{q},t)]$$

and its consequence

$$\delta q^T \frac{\partial\mathcal{H}}{\partial q} + \delta p^T \frac{\partial\mathcal{H}}{\partial p} = \delta p^T\dot{q} + p^T\delta\dot{q} - \delta q^T \frac{\partial\mathcal{L}}{\partial q} - \delta\dot{q}^T \frac{\partial\mathcal{L}}{\partial\dot{q}}$$

$$= \delta p^T\dot{q} - \delta q^T \frac{\partial\mathcal{L}}{\partial q}$$

or

$$\delta q^T\left(\frac{\partial\mathcal{H}}{\partial q} + \frac{\partial\mathcal{L}}{\partial q}\right) + \delta p^T\left(\frac{\partial\mathcal{H}}{\partial p} - \dot{q}\right) = 0 \tag{159}$$

The independence of the elements in $\delta q^T$ and $\delta p^T$ permits the conclusions

$$\frac{\partial\mathcal{L}}{\partial q} = -\frac{\partial\mathcal{H}}{\partial q} \quad \text{or} \quad \mathcal{L}_{,q} = -\mathcal{H}_{,q} \tag{160}$$

$$\dot{q} = \frac{\partial\mathcal{H}}{\partial p} \quad \text{or} \quad \dot{q} = \mathcal{H}_{,p} \tag{161}$$

Equation (158) now becomes

$$\frac{\partial\mathcal{H}}{\partial t} = -\frac{\partial\mathcal{L}}{\partial t} \quad \text{or} \quad \mathcal{H}_{,t} = -\mathcal{L}_{,t} \tag{162}$$

Substituting Eq. (160) into Eq. (154) produces

$$\dot{p} = -\mathcal{H}_{,q} + \bar{Q} - A^T\lambda \tag{163}$$

The combination of Eqs. (163) and (161) with the constraint Eqs. (135) is a complete set of $2\nu + m$ first order equations in the $2\nu + m$ unknowns $q_1, \cdots, q_\nu$, $p_1, \cdots, p_\nu, \lambda_1, \cdots, \lambda_m$. When written as a single matrix equation, they adopt the form

$$\begin{bmatrix} 0 & U & A^T \\ \hline U & 0 & 0 \\ \hline A & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{q} \\ \hline \dot{p} \\ \hline \lambda \end{Bmatrix} = \begin{Bmatrix} -\mathcal{H}_{,q} + \bar{Q} \\ \hline \mathcal{H}_{,p} \\ \hline -B \end{Bmatrix} \tag{164}$$

Note that symmetry has been retained in the coefficient matrix.

Equation (156d) permits $\mathcal{H}_{,q}$ and $\mathcal{H}_{,p}$ to be written explicitly as

$$\mathcal{H}_{,q} = -\Gamma^T_{,q}M^{-1}(p-\Gamma) + \frac{1}{2}\{(p^T - \Gamma^T)(M^{-1})_{,q_\alpha}(p-\Gamma)\} + V_{,q} - T_{0,q} \tag{165a}$$

and

$$\mathcal{H}_{,p} = M^{-1}(p - \Gamma) \tag{165b}$$

The $n \times n$ matrix $(M^{-1})_{,q_\alpha}$ presents a computational obstacle, because for problems of high dimension, matrix inversion must be accomplished numerically, and numerical inversion cannot precede partial differentiation.

The identity[5]

$$(M^{-1})_{,q_\alpha} = -M^{-1}M_{,q_\alpha}M^{-1}$$

stemming from

$$0 = \frac{\partial}{\partial q_\alpha}(MM^{-1}) = M_{,q_\alpha}M^{-1} + M(M^{-1})_{,q_\alpha}$$

permits $\mathcal{H}_{,q}$ to be written for computational purposes as

$$\mathcal{H}_{,q} = -\Gamma^T_{,q}M^{-1}(p - \Gamma) - \frac{1}{2}\{(p^T - \Gamma^T)M^{-1}M_{,q_\alpha}M^{-1}(p - \Gamma)\} + V_{,q} - T_{0,q} \tag{165c}$$

**2. Hamilton's equations for independent generalized coordinates. When all of the generalized coordinates are independent, so the Lagrange multipliers $\lambda$ in Eq. (164) are unnecessary and there are no constraint equations, Hamilton's equations adopt a much more attractive structure. With the definition**

$$x \triangleq \left[\frac{q}{p}\right]$$

**Eq. (164) can in this case be written**

$$\{\dot{x}\} = \left[\begin{array}{c|c} 0 & U \\ \hline -U & 0 \end{array}\right]\{\mathcal{H}_{,x}\} + \left\{\frac{0}{\bar{Q}}\right\} \tag{166}$$

When $\bar{Q} = 0$, so that all generalized forces in the problem are conservative, Eq. (116) adopts the strikingly simple and elegant form identified in scalar form as Hamilton's canonical equations; these equations provide the *starting point* for most of the beautiful theory of analytical mechanics. Unfortunately, this special case is applicable to only the most preliminary approximations in spacecraft attitude dynamics, because $\bar{Q}$ is required to represent passive damping and active control torques, which provide essential features of the attitude behavior of most spacecraft.

Even for $\bar{Q} \neq 0$, Eq. (166) seems appealling because the coefficient of $\dot{x}$ on the left side of this equation is unity. The numerical integration process may appear to be accelerated quite significantly by the fact that at each integration step one can find $\dot{x}$ from the right side of the equation without the necessity of removing a coefficient of $\dot{x}$ by matrix inversion or Gaussian elimination. It should be noted however that Eq. (165c) implies that $M^{-1}$ must be evaluated to obtain $\mathcal{H}_{,x}$; the noted advantage over Lagrange's equations (see Eq. (110)) is thereby lost. Moreover, if the right side of Eq. (166) involves $\dot{q}$ explicitly in $\bar{Q}$, then it becomes nec-

[5]Suggested to the writer by Prof. A. J. A. Morgan of the University of California, Los Angeles.

essary at each integration step to use Eq. (155) to obtain $\dot{q}$ in terms of $p$, and this step also involves the inversion of $M$.

It seems that the apparent advantage of applying Hamilton's canonical equation directly[6] can be realized only when $M$ can be inverted analytically and some function of $q$, $p$, and $t$ substituted explicitly for $\dot{q}$ in the Hamiltonian, but then the same inversion facilitates integration of Lagrange's equations. Examples in the sections following will make this observation more forcefully.

As an alternative to the classical derivation culminating in Eq. (163), Hamilton's equations can be derived from Lagrange's equations in state variable form with a simple coordinate transformation. This transformation is illustrated here for the special case involving independent generalized coordinates.

Comparison of Eqs. (110) and (166), with the substitution of Eq. (165b), suggests that with the transformation

$$u = M^{-1}(p - \Gamma)$$

it must be possible to obtain the bottom half of Eq. (166) from the bottom half of Eq. (110). Thus,

$$M\dot{u} = Q + T_{0,q} - M_{,t}u - \Gamma_{,t} - Gu - \{u^T J^\alpha u\}$$

must with this transformation become (with Eq. (165c))

$$\dot{p} = -\mathcal{H}_{,q} + \overline{Q} = \Gamma^T_{,q}M^{-1}(p - \Gamma) + \frac{1}{2}\{(p^T - \Gamma^T)M^{-1}M_{,q_\alpha}M^{-1}(p - \Gamma)\}$$
$$- V_{,q} + T_{0,q} + \overline{Q}$$

This transformation is simpler to establish if instead of Eq. (110) we begin with its antecedent form, Eq. (107), which with $\dot{q} \triangleq u$ becomes

$$M\dot{u} + \dot{M}u + \dot{\Gamma} = \frac{1}{2}\{u^T M_{,q_\alpha}u\} + \Gamma^T_{,q}u + T_{0,q} + Q$$

Substituting

$$u = M^{-1}(p - \Gamma)$$

and

$$\dot{u} = \frac{d}{dt}(M^{-1})(p - \Gamma) + M^{-1}(\dot{p} - \dot{\Gamma})$$
$$= -M^{-1}\dot{M}M^{-1}(p - \Gamma) + M^{-1}(\dot{p} - \dot{\Gamma})$$

[6]If instead of Hamilton's equations (Eqs. (161) and (163)) we elect to use Eqs. (154) and (155), then we don't need $\mathcal{H}(q, p, t)$ at all, and need not invert any matrix literally to obtain a complete set of equations of motion. Of course Eqs. (154) and (155) lack the most notable computational feature of Hamilton's canonical equations, which is the absence of a coefficient matrix for the highest order derivatives. In Ref. 35 the Eqs. (154) and (155) are said to be computationally superior to Lagrange's second order equations, of which they merely represent a first order form.

we find

$$-\dot{M}M^{-1}(p-\Gamma)+\dot{p}-\dot{\Gamma}+\dot{M}M^{-1}(p-\Gamma)+\dot{\Gamma}=$$

$$\frac{1}{2}\{(p^T-\Gamma^T)M^{-1}M_{,q_\alpha}M^{-1}(p-\Gamma)\}+\Gamma^T_{,q}M^{-1}(p-\Gamma)+T_{0,q}+Q$$

or with $Q=\bar{Q}-V_{,q}$,

$$\dot{p}=\frac{1}{2}\{(p^T-\Gamma^T)M^{-1}M_{,q_\alpha}M^{-1}(p-\Gamma)\}+\Gamma^T_{,q}M^{-1}(p-\Gamma)+T_{0,q}+\bar{Q}-V_{,q}$$

confirming Hamilton's equations.

# III. Application to Nonrigid Spacecraft

## A. Multiple-Rigid-Body System Models

**1. Single rigid body. In many applications it is both commonplace and reasonable to base preliminary estimates of the attitude motions of a nonrigid spacecraft on a rigid body idealization. In addition to their direct practical utility, the equations of motion of a rigid body provide a limiting-case comparison for the equations of motion for the various idealizations of flexible spacecraft to be considered here. For these reasons, we shall derive and examine equations of motion of an externally unconstrained rigid body under a force system equivalent to the force F through the mass center combined with the torque M.**

**The appropriate equations of motion will be constructed by means of the following six procedures:**

(1) **Newton-Euler equations.**

(2) Lagrange's form of D'Alembert's principle (Subsection II-A-2).

(3) Kane's quasi-coordinate equations (Subsection II-A-4).

(4) Lagrange's generalized coordinate equations (Subsection II-B-1).

(5) Lagrange's quasi-coordinate equations (Subsection II-B-3).

(6) Hamilton's equations (Subsection II-C-2).

*a. Newton–Euler equations.* By direct application of Newton's laws, Euler produced the scalar equivalent of

$$\mathbf{F}=\mathcal{M}\ddot{\mathbf{R}} \tag{167}$$

and

$$\mathbf{M}=\dot{\mathbf{H}} \tag{168}$$

with dots over vectors denoting time derivatives in an inertial reference frame, as the equations of motion of a rigid body. In terms of the vector basis established by inertially fixed unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, Eq. (167) becomes

$$F_1 = \mathcal{M}\ddot{R}_1 \tag{169a}$$

$$F_2 = \mathcal{M}\ddot{R}_2 \tag{169b}$$

$$F_3 = \mathcal{M}\ddot{R}_3 \tag{169c}$$

where $\mathbf{F} = F_1\mathbf{i}_1 + F_2\mathbf{i}_2 + F_3\mathbf{i}_3$ and $\mathbf{R} = R_1\mathbf{i}_1 + R_2\mathbf{i}_2 + R_3\mathbf{i}_3$. In terms of the vector basis established by body-fixed unit vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ paralleling principal axes of inertia for the mass center, Eq.(168) provides "Euler's equations"

$$M_1 = I_1\dot{\omega}_1 - \omega_2\omega_3 (I_2 - I_3) \tag{169d}$$

$$M_2 = I_2\dot{\omega}_2 - \omega_3\omega_1 (I_3 - I_1) \tag{169e}$$

$$M_3 = I_3\dot{\omega}_3 - \omega_1\omega_2 (I_1 - I_2) \tag{169f}$$

where $\mathbf{M} = M_1\mathbf{b}_1 + M_2\mathbf{b}_2 + M_3\mathbf{b}_3$ and the inertial angular velocity of the body is $\boldsymbol{\omega} = \omega_1\mathbf{b}_1 + \omega_2\mathbf{b}_2 + \omega_3\mathbf{b}_3$. Here $I_1$, $I_2$, $I_3$ are the mass-center principal moments of inertia of the body.

To obtain a complete description of the motion of the body, one must augment these dynamics equations by a set of kinematics equations relating $\omega_1$, $\omega_2$, $\omega_3$ to some set of orientation parameters and their time derivatives. Among the many options that might be exercised are the nine direction cosines or some subset thereof, the four Euler parameters, or a three-angle system such as the Euler angles. The question of optimal selection for computations remains controversial, and probably depends upon the problem at hand. Here we shall make a particular choice of a 1–2–3 sequence of rotations $\theta_1$, $\theta_2$, $\theta_3$ about sequentially displaced body axes (so that $\theta_1$ is a rotation of the body about $\mathbf{i}_1$ from an orientation for which $\mathbf{b}_\alpha = \mathbf{i}_\alpha$ for $\alpha = 1, 2, 3$, $\theta_2$ is a rotation about the displaced $\mathbf{b}_2$, and $\theta_3$ a rotation about the final $\mathbf{b}_3$). This is a restrictive choice, which permits the identification of these same attitude variables as independent generalized coordinates in subsequent calculations. (Such an interpretation would not be possible if we used the four Euler parameters, for example, since they are not independent.)

In terms of the attitude angles $\theta_1$, $\theta_2$, $\theta_3$, the kinematics equations are

$$\omega_1 = \dot{\theta}_1 c_2 c_3 + \dot{\theta}_2 s_3 \tag{170a}$$

$$\omega_2 = \dot{\theta}_2 c_3 - \dot{\theta}_1 c_2 s_3 \tag{170b}$$

$$\omega_3 = \dot{\theta}_3 + \dot{\theta}_1 s_2 \tag{170c}$$

or equivalently

$$\begin{bmatrix} c_2c_3 & s_3 & 0 \\ -c_2s_3 & c_3 & 0 \\ s_2 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \tag{170d}$$

where $c_\alpha \triangleq \cos\theta_\alpha$ and $s_\alpha \triangleq \sin\theta_\alpha$ for $\alpha = 2, 3$, Equation (170d) can be inverted to obtain

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} = \frac{1}{c_2} \begin{bmatrix} c_3 & -s_3 & 0 \\ c_2 s_3 & c_2 c_3 & 0 \\ -s_2 c_3 & s_2 s_3 & c_2 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \tag{171}$$

Equation (171) must be considered in conjunction with Eqs. (169) when comparisons are made with alternative formulations.

*b. Lagrange's form of D'Alembert's principle.* For a single rigid body, we can set $\mathcal{B} = 1$ and $\mathcal{P} = 0$ in Eq. (28) to obtain (ignoring Eq. (12))

$$(\mathbf{F} - \mathcal{M}\ddot{\mathbf{R}}) \cdot \partial\dot{\mathbf{R}}/\partial\dot{q}_k + (\mathbf{M} - \dot{\mathbf{H}}) \cdot \partial\omega/\partial\dot{q}_k = 0 \qquad k = 1, \cdots, 6 \tag{172}$$

With the labeling $q_\alpha = R_\alpha$ $(\alpha = 1, 2, 3)$ and $q_4 \triangleq \theta_1$, $q_5 \triangleq \theta_2$, $q_6 \triangleq \theta_3$, Eq. (172) produces

$$(\mathbf{F} - \mathcal{M}\ddot{\mathbf{R}}) \cdot \mathbf{b}_1 = 0 \tag{173a}$$

$$(\mathbf{F} - \mathcal{M}\ddot{\mathbf{R}}) \cdot \mathbf{b}_2 = 0 \tag{173b}$$

$$(\mathbf{F} - \mathcal{M}\ddot{\mathbf{R}}) \cdot \mathbf{b}_3 = 0 \tag{173c}$$

$$(\mathbf{M} - \dot{\mathbf{H}}) \cdot (\mathbf{b}_1 \cos\theta_2 \cos\theta_3 - \mathbf{b}_2 \cos\theta_2 \sin\theta_3 + \mathbf{b}_3 \sin\theta_2) = 0 \tag{173d}$$

$$(\mathbf{M} - \dot{\mathbf{H}}) \cdot (\mathbf{b}_1 \sin\theta_3 + \mathbf{b}_2 \cos\theta_3) = 0 \tag{173e}$$

$$(\mathbf{M} - \dot{\mathbf{H}}) \cdot \mathbf{b}_3 = 0 \tag{173f}$$

For Eqs. (173) to comprise a complete set, the vector $\dot{\mathbf{H}}$ must be written in terms of angles $\theta_1$, $\theta_2$, $\theta_3$ and their first and second derivatives (rather than in terms of $\omega_1$, $\omega_2$, $\omega_3$ and their first derivatives). Alternatively, one could accept in Eqs. (173) the substitutions

$$\dot{\mathbf{H}} \cdot \mathbf{b}_1 = I_1\dot{\omega}_1 - \omega_2\omega_3 (I_2 - I_3) \tag{174a}$$

$$\dot{\mathbf{H}} \cdot \mathbf{b}_2 = I_2\dot{\omega}_2 - \omega_3\omega_1 (I_3 - I_1) \tag{174b}$$

$$\dot{\mathbf{H}} \cdot \mathbf{b}_3 = I_3\dot{\omega}_3 - \omega_1\omega_2 (I_1 - I_2) \tag{174c}$$

and then augment Eqs. (173) by Eqs. (170) to obtain a complete formulation of the problem.

*c. Kane's quasi-coordinate equations.* In application to the externally unconstrained rigid body, Kane's quasi-coordinate equations (represented generally by Eqs. (66b), (55), and (64c)) are simplified by the choice

$$u_1 \triangleq \dot{R}_1 \tag{175a}$$

$$u_2 \triangleq \dot{R}_2 \tag{175b}$$

$$u_3 \triangleq \dot{R}_3 \tag{175c}$$

$$u_4 \overset{\Delta}{=} \omega_1 = \dot{\theta}_1 \cos\theta_2 \cos\theta_3 + \dot{\theta}_2 \sin\theta_3 \tag{175d}$$

$$u_5 \overset{\Delta}{=} \omega_2 = \dot{\theta}_2 \cos\theta_3 - \dot{\theta}_1 \cos\theta_2 \sin\theta_3 \tag{175e}$$

$$u_6 \overset{\Delta}{=} \omega_3 = \dot{\theta}_3 + \dot{\theta}_1 \sin\theta_2 \tag{175f}$$

The equations of motion are, for $k = 1, \cdots, 6$

$$f_k + f_k^* = 0 \tag{176}$$

where (as in Eq. (65))

$$f_k \overset{\Delta}{=} \mathbf{F} \cdot \mathbf{V}_k + \mathbf{M} \cdot \boldsymbol{\omega}_k \tag{177a}$$

and

$$f_k^* \overset{\Delta}{=} -\mathcal{M}\ddot{\mathbf{R}} \cdot \mathbf{V}_k - \dot{\mathbf{H}} \cdot \boldsymbol{\omega}_k \tag{177b}$$

The twelve vectors $\mathbf{V}_k$ and $\boldsymbol{\omega}_k$ are available by inspection of

$$\dot{\mathbf{R}} = \dot{R}_1\mathbf{i}_1 + \dot{R}_2\mathbf{i}_2 + \dot{R}_3\mathbf{i}_3 \tag{178a}$$

and

$$\boldsymbol{\omega} = \omega_1\mathbf{b}_1 + \omega_2\mathbf{b}_2 + \omega_3\mathbf{b}_3 \tag{178b}$$

when compared to the defining relationships (Eq. (65))

$$\dot{\mathbf{R}} \overset{\Delta}{=} \sum_{k=1}^{6} \mathbf{V}_k u_k + \mathbf{V}_t \tag{179a}$$

and

$$\boldsymbol{\omega} = \sum_{k=1}^{6} \boldsymbol{\omega}_k u_k + \boldsymbol{\omega}_t \tag{179b}$$

Comparison yields

$$\begin{aligned}
\mathbf{V}_1 &= \mathbf{i}_1 & \boldsymbol{\omega}_1 &= 0 \\
\mathbf{V}_2 &= \mathbf{i}_2 & \boldsymbol{\omega}_2 &= 0 \\
\mathbf{V}_3 &= \mathbf{i}_3 & \boldsymbol{\omega}_3 &= 0 \\
\mathbf{V}_4 &= 0 & \boldsymbol{\omega}_4 &= \mathbf{b}_1 \\
\mathbf{V}_5 &= 0 & \boldsymbol{\omega}_5 &= \mathbf{b}_2 \\
\mathbf{V}_6 &= 0 & \boldsymbol{\omega}_6 &= \mathbf{b}_3 \\
\mathbf{V}_t &= 0 & \boldsymbol{\omega}_t &= 0
\end{aligned} \tag{180}$$

By using material from Eq. (174) to Eq. (180), one can extract from Eq. (176) the equations of motion

$$F_1 - \mathcal{M}\ddot{R}_1 = 0 \tag{181a}$$

$$F_2 - \mathcal{M}\ddot{R}_2 = 0 \tag{181b}$$

$$F_3 - \mathcal{M}\ddot{R}_3 = 0 \tag{181c}$$

$$M_1 - [I_1\dot{\omega}_1 - \omega_2\omega_3 (I_2 - I_3)] = 0 \tag{181d}$$

$$M_2 - [I_2\dot{\omega}_2 - \omega_3\omega_1 (I_3 - I_1)] = 0 \tag{181e}$$

$$M_3 - [I_3\dot{\omega}_3 - \omega_1\omega_2 (I_1 - I_2)] = 0 \tag{181f}$$

Equations (181) and (175d through f) comprise a complete set, which is obviously identical to Eqs. (169) and (170) obtained directly from the Newton-Euler vector equations.

*d. Lagrange's generalized coordinate equations.* As noted in Subsection II-B-1, Lagrange's equations for independent generalized coordinates (see Eq. (70)) are identical in final form to the results of the application of D'Alembert's principle (see Eq. (67), or for a single rigid body, Eq. (172)). With this Lagrangian approach we are committed to the exclusive use of generalized coordinates $q_1, \cdots, q_6$, so our results will not match those obtained by substituting Eq. (174) into Eq. (173), but will correspond to those obtained by combining Eqs. (170), (174), and (173). Of course Lagrange would not have us proceed in this fashion. Instead we should calculate

$$\begin{aligned}
T &= \frac{1}{2}\mathcal{M}\dot{\mathbf{R}}\cdot\dot{\mathbf{R}} + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega} \\
&= \frac{\mathcal{M}}{2}[\dot{R}_1^2 + \dot{R}_2^2 + \dot{R}_3^2] + \frac{1}{2}[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2] \\
&= \frac{\mathcal{M}}{2}(\dot{R}_1^2 + \dot{R}_2^2 + \dot{R}_3^2) + \frac{1}{2}I_1(\dot{\theta}_1 \cos\theta_2 \cos\theta_3 + \dot{\theta}_2 \sin\theta_3)^2 \\
&\quad + \frac{1}{2}I_2(\dot{\theta}_2\cos\theta_3 - \dot{\theta}_1 \cos\theta_2 \sin\theta_3)^2 + \frac{1}{2}I_3(\dot{\theta}_3 + \dot{\theta}_1 \sin\theta_2)^2
\end{aligned} \tag{182a}$$

and then construct the partial derivatives

$$\begin{aligned}
&\frac{\partial T}{\partial \dot{R}_1} = \mathcal{M}\dot{R}_1, \qquad \frac{\partial T}{\partial \dot{R}_2} = \mathcal{M}\dot{R}_2, \qquad \frac{\partial T}{\partial \dot{R}_3} = \mathcal{M}\dot{R}_3 \\
&\frac{\partial T}{\partial \dot{\theta}_1} = I_1 c_2 c_3 (\dot{\theta}_1 c_2 c_3 + \dot{\theta}_2 s_3) - I_2 c_2 s_3 (\dot{\theta}_2 c_3 - \dot{\theta}_1 c_2 s_3) \\
&\qquad\quad + I_3 s_2 (\dot{\theta}_3 + \dot{\theta}_1 s_2) \\
&\frac{\partial T}{\partial \dot{\theta}_2} = I_1 s_3 (\dot{\theta}_1 c_2 c_3 + \dot{\theta}_2 s_3) + I_2 c_3 (\dot{\theta}_2 c_3 - \dot{\theta}_1 c_2 s_3) \\
&\frac{\partial T}{\partial \dot{\theta}_3} = I_3 (\dot{\theta}_3 + \dot{\theta}_1 s_2)
\end{aligned} \tag{182b}$$

and the total derivatives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{R}_1}\right) = \mathcal{M}\ddot{R}_1, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{R}_2}\right) = \mathcal{M}\ddot{R}_2, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{R}_3}\right) = \mathcal{M}\ddot{R}_3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_1}\right) = \ddot\theta_1 (I_1 c_2^2 c_3^2 + I_2 c_2^2 s_3^2 + I_3 s_2^2)$$
$$+ \ddot\theta_2 c_2 s_3 c_3 (I_1 - I_2) + \ddot\theta_3 s_2 I_3 + \dot\theta_2^2 s_2 s_3 c_3 (I_2 - I_1)$$
$$+ \dot\theta_1 \dot\theta_2\, 2 s_2 c_2 (I_3 - I_1 c_3^2 - I_2 s_3^2)$$
$$+ \dot\theta_2 \dot\theta_3 [I_3 c_2 + (I_1 - I_2) c_2 (c_3^2 - s_3^2)]$$
$$+ 2\dot\theta_3 \dot\theta_1 c_2^2 s_3 c_3 (I_2 - I_1)$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_2}\right) = \ddot\theta_1 c_2 s_3 c_3 (I_1 - I_2) + \ddot\theta_2 (I_1 s_3^2 + I_2 c_3^2)$$
$$+ \dot\theta_1 \dot\theta_2 s_3 c_3 s_2 (I_2 - I_1) + 2\dot\theta_2 \dot\theta_3 s_3 c_3 (I_1 - I_2)$$
$$+ \dot\theta_3 \dot\theta_1 c_2 (c_3^2 - s_3^2)(I_1 - I_2)$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_3}\right) = I_3 (\ddot\theta_3 + \ddot\theta_1 s_2 + \dot\theta_1 \dot\theta_2 c_2) \tag{182c}$$

where $c_\alpha \triangleq \cos\theta_\alpha$ and $s_\alpha \triangleq \sin\theta_\alpha$, $\alpha = 1, 2, 3$.

Lagrange's equations also require the partial derivatives

$$\frac{\partial T}{\partial R_1} = 0, \qquad \frac{\partial T}{\partial R_2} = 0, \qquad \frac{\partial T}{\partial R_3} = 0$$
$$\frac{\partial T}{\partial \theta_1} = 0$$
$$\frac{\partial T}{\partial \theta_2} = -I_1 (\dot\theta_1 c_2 c_3 + \dot\theta_2 s_3) s_2 \dot\theta_1 c_3 + I_2 (\dot\theta_2 c_3 - \dot\theta_1 c_2 s_3) \dot\theta_1 s_2 s_3$$
$$+ I_3 (\dot\theta_3 + \dot\theta_1 s_2) \dot\theta_1 c_2$$
$$\frac{\partial T}{\partial \theta_3} = I_1 (\dot\theta_1 c_2 c_3 + \dot\theta_2 s_3)(\dot\theta_2 c_3 - \dot\theta_1 c_2 s_3) - I_2 (\dot\theta_2 c_3 - \dot\theta_1 c_2 s_3)(\dot\theta_2 s_3 + \dot\theta_1 c_2 c_3) \tag{182d}$$

Finally, Lagrange's equations require the generalized coordinates $Q_k$, $k = 1, \cdots, 6$, as given by

$$Q_1 = \mathbf{F}\cdot\frac{\partial \dot{\mathbf{R}}}{\partial \dot R_1} + \mathbf{M}\cdot\frac{\partial \boldsymbol\omega}{\partial \dot R_1} = \mathbf{F}\cdot\mathbf{i}_1 = F_1$$
$$Q_2 = \mathbf{F}\cdot\frac{\partial \dot{\mathbf{R}}}{\partial \dot R_2} + \mathbf{M}\cdot\frac{\partial \boldsymbol\omega}{\partial \dot R_2} = \mathbf{F}\cdot\mathbf{i}_2 = F_2$$
$$Q_3 = \mathbf{F}\cdot\frac{\partial \dot{\mathbf{R}}}{\partial \dot R_3} + \mathbf{M}\cdot\frac{\partial \boldsymbol\omega}{\partial \dot R_3} = \mathbf{F}\cdot\mathbf{i}_3 = F_3$$
$$Q_4 = \mathbf{F}\cdot\frac{\partial \dot{\mathbf{R}}}{\partial \dot\theta_1} + \mathbf{M}\cdot\frac{\partial \boldsymbol\omega}{\partial \dot\theta_1} = \mathbf{M}\cdot(\mathbf{b}_1 c_2 c_3 - \mathbf{b}_2 c_2 s_3 + \mathbf{b}_3 s_2)$$
$$= M_1 c_2 c_3 - M_2 c_2 s_3 + M_3 s_2$$
$$Q_5 = \mathbf{F}\cdot\frac{\partial \dot{\mathbf{R}}}{\partial \dot\theta_2} + \mathbf{M}\cdot\frac{\partial \boldsymbol\omega}{\partial \dot\theta_2} = \mathbf{M}\cdot(\mathbf{b}_1 s_3 + \mathbf{b}_2 c_3) = M_1 s_3 + M_2 c_3$$
$$Q_6 = \mathbf{F}\cdot\frac{\partial \dot{\mathbf{R}}}{\partial \dot\theta_3} + \mathbf{M}\cdot\frac{\partial \boldsymbol\omega}{\partial \dot\theta_3} = \mathbf{M}\cdot\mathbf{b}_3 = M_3 \tag{182e}$$

Substituting these results into Lagrange's equations in the form of Eq. (70) produces the following equations of motion for the single rigid body:

$$F_1 = \mathcal{M}\ddot{R}_1, \qquad F_2 = \mathcal{M}\ddot{R}_2, \qquad F_3 = \mathcal{M}\ddot{R}_3$$

$$\begin{aligned}
&\ddot{\theta}_1 \left[c_2^2 (I_1 c_3^2 + I_2 s_3^2) + I_3 s_2^2\right] + \ddot{\theta}_2 c_2 s_3 c_3 (I_1 - I_2) + \ddot{\theta}_3 s_2 I_3 + \dot{\theta}_2^2 s_2 s_3 c_3 (I_2 - I_1) \\
&\quad + 2\dot{\theta}_1 \dot{\theta}_2 s_2 c_2 (I_3 - I_1 c_3^2 - I_2 s_3^2) + \dot{\theta}_2 \dot{\theta}_3 c_2 \left[I_3 + (I_1 - I_2)(c_3^2 - s_3^2)\right] \\
&\quad + 2\dot{\theta}_3 \dot{\theta}_1 c_2^2 s_3 c_3 (I_2 - I_1) = M_1 c_2 c_3 - M_2 c_2 s_3 + M_3 s_2 \\
&\ddot{\theta}_1 c_2 s_3 c_3 (I_1 - I_2) + \ddot{\theta}_2 (I_1 s_3^2 + I_2 c_3^2) + 2\dot{\theta}_2 \dot{\theta}_3 s_3 c_3 (I_1 - I_2) \\
&\quad + \dot{\theta}_3 \dot{\theta}_1 c_2 \left[-I_3 + (I_1 - I_2)(c_3^2 - s_3^2)\right] - \dot{\theta}_1^2 s_2 c_2 \left[I_3 - I_1 c_3^2 - I_2 s_3^2\right] \\
&\quad = M_1 s_3 + M_2 c_3
\end{aligned}$$

$$\begin{aligned}
I_3 \ddot{\theta}_1 s_2 + I_3 \ddot{\theta}_3 + \dot{\theta}_1 \dot{\theta}_2 c_2 &\left[I_3 - (I_1 - I_2)(c_3^2 - s_3^2)\right] \\
&+ \dot{\theta}_1^2 c_2^2 s_3 c_3 (I_1 - I_2) + \dot{\theta}_2^2 s_3 c_3 (I_2 - I_1) = M_3
\end{aligned} \tag{183}$$

As noted previously, these equations are precisely what would emerge from the substitution of Eqs. (174) and (170) into Eq. (173), as indicated by D'Alembert's principle.

For computational convenience it may be desirable to rewrite Eq. (183) in first order form, following the general pattern established by Eq. (110). It then becomes apparent that the dependence of the coefficient matrix $M$ on time-varying quantities presents an obstacle to efficient computation, because this matrix must be inverted (or recourse must be taken to a Gaussian elimination) at every step of the numerical integration, in order to find $\ddot{q}$ at each time step from the available values of $\dot{q}$ and $q$. If these equations were to be used extensively and a high premium was attached to digital computer time, one could invert $M$ literally (nonnumerically) in advance of integration. Since only the lower $3 \times 3$ block of $M$ is populated off of the main diagonal, this is not a major undertaking for this little problem. In general, however, this literal inversion is not feasible for spacecraft simulation problems.

*e. Lagrange's quasi-coordinate equations.* The Lagrangian quasi-coordinate equations are presented in their most general form as Eq. (147). For the holonomic system under consideration, the term $\lambda$ involving the Lagrange multipliers is absent. By choosing the quasi-coordinate derivatives as in Eq. (175), and limiting application to a single rigid body, we can greatly reduce the complexity of the equations of motion. By comparing Eqs. (175) and (134), we find that

$$w = 0$$

and

$$W^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 c_3 & s_3 & 0 \\ 0 & 0 & 0 & c_2 s_3 & c_3 & 0 \\ 0 & 0 & 0 & s_2 & 0 & 1 \end{bmatrix} \tag{184}$$

where $s_\alpha \triangleq \sin\theta_\alpha$ and $c_\alpha \triangleq \cos\theta_\alpha$ for $\alpha = 1, 2, 3$. Thus in Eq. (147), we have

$W_{,t} = 0$. These considerations alone reduce Eq. (147a) to the simpler form

$$W\frac{d}{dt}(\overline{T}_{,u}) + [u^T W^{-1} W_{ij,q}]\,\overline{T}_{,u} - \{u^T W^{-1} W_{,q_k} \overline{T}_{,u}\} - \overline{T}_{,q} = Q$$

Equation (182a) provides

$$\overline{T} = \frac{1}{2}\mathcal{M}(u_1^2 + u_2^2 + u_3^2) + \frac{1}{2}I_1 u_4^2 + \frac{1}{2}I_2 u_5^2 + \frac{1}{2}I_3 u_6^2 \tag{185}$$

$\overline{T}_{,q}$ is eliminated for this application, and $\overline{T}_{,u}$ is provided:

$$\overline{T}_{,u} = \{\mathcal{M}u_1\ \mathcal{M}u_2\ \mathcal{M}u_3\ I_1u_4\ I_2u_5\ I_3u_6\}^T$$

$$= \left[\begin{array}{c|c} \mathcal{M}U & 0 \\ \hline 0 & I \end{array}\right]\{u\} \triangleq M'u \tag{186}$$

where $U$ is the unit matrix and $I$ is the diagonal matrix of principal moments of inertia for the mass center. The matrix equation of motion can then be represented as

$$\frac{d}{dt}\overline{T}_{,u} + W^{-1}[u^T W^{-1} W_{ij,q}]\,\overline{T}_{,u} - W^{-1}\{u^T W^{-1} W_{,q_k} \overline{T}_{,u}\} = W^{-1}Q \tag{187a}$$

or (see Eq. (147b)) as

$$\frac{d}{dt}(\overline{T}_{,u}) + W^{-1}\gamma \overline{T}_{,u} = W^{-1}Q \tag{187b}$$

With Eq. (186), the first term in Eq. (187) is trivial to record. The term on the right of Eq. (187) is also simple, since for the given choice of quasi-coordinate derivatives the generalized forces in $Q$ may be written as

$$Q_j \triangleq \frac{\partial \dot{\mathbf{R}}}{\partial \dot{q}_j}\cdot\mathbf{F} + \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_j}\cdot\mathbf{M}$$

$$= \frac{\partial}{\partial \dot{q}_j}\{u_1u_2u_3u_4u_5u_6\}\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix}$$

so that

$$Q = \frac{\partial u^T}{\partial \dot{q}}\begin{Bmatrix} F \\ \hline M \end{Bmatrix} = u^T_{,\dot{q}}\begin{Bmatrix} F \\ \hline M \end{Bmatrix} \tag{188}$$

with $F \triangleq \{F_1F_2F_3\}^T$ and $M \triangleq \{M_1M_2M_3\}^T$. Equation (134d) now combines with Eq. (188) to permit the interpretation

$$W^{-1}Q = W^{-1}W\begin{Bmatrix} F \\ \hline M \end{Bmatrix} = \begin{Bmatrix} F \\ \hline M \end{Bmatrix} \tag{189}$$

There remains for disposition in Eq. (187) the term

$$W^{-1}\gamma \overline{T}_{,u} = W^{-1}\,[u^T W^{-1} W_{ij,q}]\,\overline{T}_{,u} - W^{-1}\,\{u^T W^{-1} W_{,q_k} \overline{T}_{,u}\} \tag{190}$$

For the special case of $W$ implied by Eq. (184), explicit inversion is not difficult, and we can find

$$W^{-1} = \frac{1}{c_2}\begin{bmatrix} c_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 & c_2 s_3 & -s_2 c_3 \\ 0 & 0 & 0 & -s_3 & c_2 c_3 & s_2 s_3 \\ 0 & 0 & 0 & 0 & 0 & c_2 \end{bmatrix} \tag{191}$$

The basic ingredients of the six by six matrix $\gamma$ in Eq. (190) are the six $6 \times 6$ matrices $W_{,q_k}$ and the thirty-six $6 \times 1$ matrices $W_{ij,q}$. The first set of six matrices can be obtained explicitly from Eq. (184) after transposition and partial differentiation as follows:

$W_{,q_k}$ **is the null matrix for** $k = 1, 2, 3, 4$.

$W_{,q_5} \triangleq W_{,\theta_2}$ **is null except for the three elements**

$$[W_{,q_5}]_{44} = -s_2 c_3$$
$$[W_{,q_5}]_{45} = s_2 s_3$$
$$[W_{,q_5}]_{46} = c_2$$

$W_{,q_6} = W_{,\theta_3}$ **is null except for the four elements**

$$[W_{,q_6}]_{44} = -c_2 s_3$$
$$[W_{,q_6}]_{45} = -c_2 c_3$$
$$[W_{,q_6}]_{54} = c_3$$
$$[W_{,q_6}]_{55} = -s_3$$

The second set of 36 column matrices $W_{ij,q}$ is available by transposition of $W^T$ in Eq. (183) and partial differentiation of its elements with respect to $q$. The results are all columns of zeros except for $\{W_{44,q}\}$, $\{W_{54,q}\}$, $\{W_{45,q}\}$, $\{W_{55,q}\}$, and $\{W_{46,q}\}$, which are null except for the following nonzero elements:

$$\begin{aligned} &\{W_{44,q}\}_5 = -s_2 c_3 \qquad && \{W_{44,q}\}_6 = -c_2 s_3 \\ &\{W_{54,q}\}_6 = c_3 && \\ &\{W_{45,q}\}_5 = s_2 s_3 && \{W_{45,q}\}_6 = -c_2 c_3 \\ &\{W_{55,q}\}_6 = -s_3 && \\ &\{W_{46,q}\}_5 = c_2 && \end{aligned} \tag{193}$$

As indicated by Eq. (190), the construction of $\gamma$ requires the matrix product (from Eqs. (175) and (192)):

$$\begin{aligned} u^T W^{-1} = \{u_1 \quad u_2 \quad u_3 \quad & (u_4 c_3 - u_5 s_3)/c_2 \quad (u_4 s_3 + u_5 c_3) \\ & (-u_4 s_2 c_3 + u_5 s_2 s_3 + u_6 c_2)/c_2\} \end{aligned} \tag{194}$$

Equation (190) can now be used to construct the following expression for the $r$th row $\gamma^{(r)}$ of the matrix $\gamma$ as follows:

$$\gamma^{(r)} = u^T W^{-1} W_{rj,q} - u^T W^{-1} W_{,q_r} = u^T W^{-1} (W_{rj,q} - W_{,q_r}) \tag{195}$$

Thus

$$\gamma^{(i)} = u^T W^{-1} (W_{ij,q} - W_{,q_i}) = 0, \qquad \text{for} \qquad i = 1, 2, 3 \tag{196a}$$

$$\gamma^{(4)} = u^T W^{-1} (W_{4j,q} - W_{,q_4}) = u^T W^{-1} W_{4j,q}$$

or

$$\gamma^{(4)} = u^T W^{-1} \quad [0 \mid 0 \mid 0 \mid W_{44,q} \mid W_{45,q} \mid W_{46,q}]$$

or

$$\gamma^{(4)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ (u_4 s_3 + u_5 c_3)(-s_2 c_3) + (-u_4 s_2 c_3 + u_5 s_2 s_3 + u_6 c_2)(-s_3) \\ (u_4 s_3 + u_5 c_3) s_2 s_3 + (-u_4 s_2 c_3 + u_5 s_2 s_3 + u_6 c_2)(-c_3) \\ (u_4 s_3 + u_5 c_3) c_2 \end{Bmatrix}^T$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -u_5 s_2 - u_6 c_2 s_3 \\ u_4 s_2 - u_6 c_2 c_3 \\ u_4 c_2 s_3 + u_5 c_2 c_3 \end{Bmatrix}^T \tag{196b}$$

and similarly for $r = s$, except that $W_{,q_5} \neq 0$, so that

$$\gamma^{(5)} = u^T W^{-1} (W_{5j,q} - W_{,q_5})$$

$$= u^T W^{-1} [0 \mid 0 \mid 0 \mid W_{54,q} \mid W_{55,q} \mid 0] - u^T W^{-1} W_{,q_5}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ [(-u_4 s_2 c_3 + u_5 s_2 s_3 + u_6 c_2)/c_2]\, c_3 - [(u_4 c_3 - u_5 s_3)/c_2]\,(-s_2 c_3) \\ [(-u_4 s_2 c_3 + u_5 s_2 s_3 + u_6 c_2)/c_2]\,(-s_3) - [(u_4 c_3 - u_5 s_3)/c_2]\,(s_2 s_3) \\ -[(u_4 c_3 - u_5 s_3)/c_2]\, c_2 \end{Bmatrix}^T$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_6 c_3 \\ -u_6 s_3 \\ -u_4 c_3 + u_5 s_3 \end{Bmatrix}^T \tag{196c}$$

The sixth row of $\gamma$ is more easily found to be given by

$$\gamma^{(6)} = u^T W^{-1} (W_{6j,q} - W_{,q_6}) = -u^T W^{-1} W_{,q_6} = \{0 \; 0 \; 0 \quad -u_5 \quad u_4 \quad 0\} \tag{196d}$$

Thus the $6 \times 6$ matrix $\gamma$ can be written in terms of $3 \times 3$ partitions as

$$\gamma = \left[\begin{array}{c|ccc} 0 & & 0 & \\ \hline & -u_5 s_2 - u_6 c_2 s_3 & u_4 s_2 - u_6 c_2 c_3 & u_4 c_2 s_3 + u_5 c_2 c_3 \\ 0 & u_6 c_3 & -u_6 s_3 & -u_4 c_3 + u_5 s_3 \\ & -u_5 & u_4 & 0 \end{array}\right] \tag{197}$$

The equations of motion (see Eq. (187b)) require the matrix product $W^{-1}\gamma$, which from Eqs. (191) and (197) acquires the remarkably simple form (written in terms of $3 \times 3$ partitions)

$$W^{-1}\gamma = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \tilde{\omega} \end{array}\right] \tag{198}$$

where, as in item 14 of Appendix A, noting Eq. (175),

$$\tilde{\omega} \stackrel{\Delta}{=} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -u_6 & u_5 \\ u_6 & 0 & -u_4 \\ -u_5 & u_4 & 0 \end{bmatrix} \tag{199}$$

It finally becomes apparent upon substituting Eqs. (198), (188), and (186) into Eq. (187b) that the quasi-coordinate equations of motion are really very simple, since they reduce to

$$\left[\begin{array}{c|c} \mathcal{M}U & 0 \\ \hline 0 & I \end{array}\right] \{\dot{u}\} + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \tilde{\omega} \end{array}\right] \left[\begin{array}{c|c} \mathcal{M}U & 0 \\ \hline 0 & I \end{array}\right] \{u\} = \begin{Bmatrix} F \\ M \end{Bmatrix} \tag{200}$$

By the definitions of the elements of $\{u\}$ in Eq. (175), this result is precisely the same as that obtained trivially by recording Newton's second law and Euler's equations (see Eq. (169)). The same results were also obtained in Eq. (181), using Kane's quasi-coordinate formulation.

*f. Hamilton's equations.* The sixth and final equation formulation procedure to be applied here to the single rigid body is that due to Hamilton, as represented in matrix form by Eq. (166).

Hamilton's equations involve as unknowns the generalized coordinates previously employed in Lagrange's equations, and the generalized momenta in the matrix $p$ defined in Eq. (153). In this application the potential energy $V$ is taken as zero, so that all of the external forces and torques are represented by the generalized forces in the matrix $Q$, which then replaces $\overline{Q}$ in Eq. (166). Since the Lagrangian is here identical to the kinetic energy, the generalized momentum matrix in Eq. (153) becomes

$$p = T_{,\dot{q}} \tag{201a}$$

(Actually this is an expression of completely general validity, since $V_{,\dot{q}} = 0$).

As indicated by Eq. (105), $T$ is a second degree form in the generalized velocities, so that $p$ is a linear form in $\dot{q}$, and Eq. (201) can be written as

$$p = M\dot{q} + \Gamma \tag{201b}$$

for a square symmetric matrix $M$ and a column matrix $\Gamma$ defined by Eqs. (106) and (107). (Comparison with Eq. (134) indicates that $p$ qualifies as a quasi-coordinate derivative matrix, but $p$ has special properties that stem from the definition in Eq. (201a).)

Physical considerations guarantee that Eq. (201b) can be inverted to obtain

$$\dot{q} = M^{-1}(p - \Gamma) \tag{202}$$

The application of Hamilton's equations to the rigid body involves the following steps:

(1) Construct $T = T(q, \dot{q}, t)$ from basic definitions.

(2) Construct $p$ from Eq. (201a), and thereby identify $M$.

(3) Find $M^{-1}$, and record Eq. (202).

(4) Use Eq. (202) to obtain

$$T(q, \dot{q}, t) = \overline{T}(q, p, t) \tag{203}$$

(5) Construct the Hamiltonian from its definition in Eq. (156), which here reduces to

$$\begin{aligned} \mathcal{H} &= p^T\dot{q} - T \\ &= p^T M^{-1} p - p^T M^{-1}\Gamma - \overline{T} \end{aligned} \tag{204}$$

Note that $\mathcal{H} = \mathcal{H}(q, p, t)$ must be obtained before proceeding.

(6) Find the partial derivatives $\mathcal{H}_{,q}$ and $\mathcal{H}_{,p}$ that comprise $\mathcal{H}_{,x}$ in Eq. (166).

(7) Find $\overline{Q}$ (here identical to $Q$) from Eq. (188) as

$$Q = u^T_{,\dot{q}} \left\{ \frac{F}{M} \right\} \tag{205}$$

where $u = \{\dot{q}_1 \quad \dot{q}_2 \quad \dot{q}_3 \quad \omega_1 \quad \omega_2 \quad \omega_3\}^T$. By substituting for $\dot{q}$ from Eq. (202) into Eq. (205) one can obtain $Q = Q(q, p, t)$, and incorporate the result into Eq. (166).

Our proposed application to the single rigid body is simplified by the fact that in constructing Lagrange's equations we have already found $T$ and $Q$ in terms of $\dot{q}$, $q$, and $t$ (see Eqs. (182a) and (182e)). Thus we can easily find from Eqs. (201a) and (182b) the scalar generalized momenta

$$p_1 = \mathcal{M}\dot{R}_1, \qquad p_2 = \mathcal{M}\dot{R}_2, \qquad p_3 = \mathcal{M}\dot{R}_3$$

$$p_4 = \frac{\partial T}{\partial \dot{\theta}_1} = \dot{\theta}_1 \left[ I_3 s_2^2 + c_2^2 (I_1 c_3^2 + I_2 s_3^2) \right] + \dot{\theta}_2 (I_1 - I_2) c_2 s_3 c_3 + \dot{\theta}_3 I_3 s_2$$

$$p_5 = \frac{\partial T}{\partial \dot{\theta}_2} = \dot{\theta}_1 (I_1 - I_2)\, c_2 s_3 c_3 + \dot{\theta}_2 (I_1 s_3^2 + I_2 c_3^2)$$

$$p_6 = \frac{\partial T}{\partial \dot{\theta}_3} = \dot{\theta}_1 I_3 s_2 + \dot{\theta}_3 I_3 \tag{206}$$

Thus in Eq. (201b) the matrix $\Gamma$ is zero and $M$ is given in terms of $3 \times 3$ partitions by

$$M = \left[\begin{array}{c|ccc} U & & 0 & \\ \hline & [I_3 s_2^2 + c_2^2 (I_1 c_3^2 + I_2 s_3^2)] & (I_1 - I_2)\, c_2 s_3 c_3 & I_3 s_2 \\ 0 & (I_1 - I_2)\, c_2 s_3 c_3 & I_1 s_3^2 + I_2 c_3^2 & 0 \\ & I_3 s_2 & 0 & I_3 \end{array}\right] \tag{207}$$

The next step called for in the outline is the construction of $M^{-1}$, as required for the construction of $\overline{T}(q, p, t)$ and $\mathcal{H}$. This promises to be a tedious chore even for this simple example, and it would be a major undertaking for a spacecraft model of any complexity.[7] This matrix inversion should be accomplished literally (not numerically) whenever possible, since it is used in the construction of $\mathcal{H}$, which must be subjected to partial differentiation with respect to $q$, which appears in $M$. The only alternative for digital computation is the use of the identity

$$M^{-1}{}_{,q_\alpha} = -M^{-1} M_{,q_\alpha} M^{-1}$$

as noted previously.

For the special case of the rigid body, we can circumvent the problems of inverting $M$ and constructing $\mathcal{H}$, and pursue a somewhat easier path that makes intermediate use of the angular velocity components $\omega_1, \omega_2, \omega_3$. Rather than use Eq. (203) to obtain $\mathcal{H}$, we can return to the representation found in Eq. (156c), noting that in this case $V = 0$ and $T = T_2$ (so $T_0 = 0$). Therefore $\mathcal{H}$ is identical to $\overline{T}$, and we can skip the step indicated by Eq. (204).

Moreover, rather than apply Eq. (203) to find $\overline{T}(q, p, t)$ from $T(q, \dot{q}, t)$ by substituting Eq. (202) for $\dot{q}$, we can proceed directly to $\overline{T}(q, p, t)$ from kinetic energy in the form

$$T = \frac{\mathcal{M}}{2}(\dot{R}_1^2 + \dot{R}_2^2 + \dot{R}_3^2) + \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \tag{208}$$

The simplicity of the expressions for $p_1$, $p_2$, and $p_3$ in Eq. (206) permits the representation

$$T = \frac{1}{2\mathcal{M}}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \tag{209}$$

so that we require expressions for $\omega_1$, $\omega_2$, and $\omega_3$ in terms of $q$ and $p$ if we are to obtain $\overline{T}(q, p, t)$. Comparison of Eqs. (206) and (170) leads to the relationships

$$p_4 = I_1 c_2 c_3 \omega_1 - I_2 c_2 s_3 \omega_2 + I_3 s_2 \omega_3 \tag{210a}$$

[7]Note again that the same matrix $M$ emerges as a coefficient of the most highly differentiated terms in Lagrange's equations. See Eq. (110) and Eq. (183).

$$p_5 = I_1 s_3 \omega_1 + I_2 c_3 \omega_2 \tag{210b}$$

$$p_6 = I_3 \omega_3 \tag{210c}$$

or equivalently[8]

$$\begin{Bmatrix} p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{Bmatrix} \tag{210d}$$

This relationship is somewhat more easily inverted than is $M$ in Eq. (207); the result shows

$$\begin{Bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{Bmatrix} = \frac{1}{c_2}\begin{bmatrix} c_3 & c_2 s_3 & -s_2 c_3 \\ -s_3 & c_2 c_3 & s_2 s_3 \\ 0 & 0 & c_2 \end{bmatrix} \begin{Bmatrix} p_4 \\ p_5 \\ p_6 \end{Bmatrix} \tag{211}$$

Substitution into Eq. (209) now produces

$$\begin{aligned} \bar{T} &= \frac{1}{2\mathcal{M}}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2I_1 c_2^2}(p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3)^2 \\ &\quad + \frac{1}{2I_2 c_2^2}(-p_4 s_3 + p_5 c_2 c_3 + p_6 s_2 c_3)^2 + \frac{1}{2I_3} p_6^2 = \mathcal{H} \end{aligned} \tag{212}$$

Partial differentiation now yields

$$\mathcal{H}_{,p_1} = p_1/\mathcal{M}, \qquad \mathcal{H}_{,p_2} = p_2/\mathcal{M}, \qquad \mathcal{H}_{,p_3} = p_3/\mathcal{M}$$

$$\mathcal{H}_{,p_4} = \frac{c_3}{I_1 c_2^2}(p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3) - \frac{s_3}{I_2 c_2^2}(-p_4 s_3 + p_5 c_2 c_3 + p_6 s_2 s_3)$$

$$\mathcal{H}_{,p_5} = \frac{s_3}{I_1 c_2}(p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3) + \frac{c_3}{I_2 c_2}(-p_4 s_3 + p_5 c_2 c_3 + p_6 s_2 s_3)$$

$$\mathcal{H}_{,p_6} = \frac{-s_2 c_3}{I_1 c_2^2}(p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3) + \frac{s_2 s_3}{I_2 c_2^2}(-p_4 s_3 + p_5 c_2 c_3 + p_6 s_2 s_3) + p_6/I_3$$

$$\mathcal{H}_{,q_1} = 0, \qquad \mathcal{H}_{,q_2} = 0, \qquad \mathcal{H}_{,q_3} = 0, \qquad \mathcal{H}_{,q_4} = 0$$

---

[8]Note that the coefficient matrix in Eq. (210d) is contained in $W$ as defined by Eq. (184). This relationship can be displayed even more dramatically by absorbing Eq. (210d) in the more comprehensive statement

$$p = WM'u \tag{210e}$$

where the elements of $u$ are defined by Eq. (175) and $M'$ is defined in terms of $3 \times 3$ partitions as

$$M' \triangleq \left[\begin{array}{c|c} \mathcal{M}U & 0 \\ \hline 0 & I \end{array}\right]$$

with $U$ the unit matrix and $I$ the diagonal matrix of principal moments of inertia. The combination of Eqs. (210e) and (133b) then provides

$$p = WM'W^T\dot{q} \tag{210f}$$

which when compared to Eq. (201b) yields the interesting relationship

$$M = WM'W^T \tag{210g}$$

$$\mathcal{H}_{,q_5} = \frac{-1}{I_1 c_2^2}(p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3)(p_5 s_2 s_3 + p_6 c_2 c_3)$$

$$+ \frac{s_2}{I_1 c_2^3}(p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3)^2$$

$$+ \frac{1}{I_2 c_2^2}(-p_4 s_3 + p_5 c_2 c_3 + p_6 s_2 s_3)(p_6 c_2 s_3 - p_5 s_2 c_3)$$

$$+ \frac{s_2}{I_2 c_2^3}(-p_4 s_3 + p_5 c_2 c_3 + p_6 s_2 s_3)^2$$

$$\mathcal{H}_{,q_6} = \frac{1}{I_1 c_2^2}(p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3)(-p_4 s_3 + p_5 c_2 c_3 + p_6 s_2 s_3)$$

$$+ \frac{1}{I_2 c_2^2}(-p_4 s_3 + p_5 c_2 c_3 + p_6 s_2 s_3)(-p_4 c_3 - p_5 c_2 s_3 + p_6 s_2 c_3) \tag{213}$$

The equations of motion are finally available by substituting Eq. (213) for the elements of $\mathcal{H}_{,q}$ in Eq. (166), and substituting Eq. (182e) for the elements of $\bar{Q}$ in that equation. The result may be written as the following system of scalar equations:

$$\dot{q}_1 = p_1/\mathcal{M} \tag{214a}$$

$$\dot{q}_2 = p_2/\mathcal{M} \tag{214b}$$

$$\dot{q}_3 = p_3/\mathcal{M} \tag{214c}$$

$$\dot{q}_4 = (p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3) c_3/(I_1 c_2^2) + (p_4 s_3 - p_5 c_2 c_3 - p_6 s_2 s_3) s_3/(I_2 c_2^2) \tag{214d}$$

$$\dot{q}_5 = (p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3) s_3/(I_1 c_2) - (p_4 s_3 - p_5 c_2 c_3 - p_6 s_2 s_3) c_3/(I_2 c_2) \tag{214e}$$

$$\dot{q}_6 = (p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3) s_2 c_3/(I_1 c_2^2)$$
$$+ (-p_4 s_3 + p_5 c_2 c_3 + p_6 s_2 s_3) s_2 s_3/(I_2 c_2^2) + p_6/I_3 \tag{214f}$$

$$\dot{p}_1 = Q_1 = F_1 \tag{214g}$$

$$\dot{p}_2 = Q_2 = F_2 \tag{214h}$$

$$\dot{p}_3 = Q_3 = F_3 \tag{214i}$$

$$\dot{p}_4 = Q_4 = M_1 c_2 c_3 - M_2 c_2 s_3 + M_3 s_2 \tag{214j}$$

$$\dot{p}_5 = (p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3)(p_5 s_2 s_3 + p_6 c_2 c_3)/(I_1 c_2^2)$$
$$- (p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3)^2 s_2/(I_1 c_2^3)$$
$$+ (p_4 s_3 - p_5 c_2 c_3 - p_6 s_2 s_3)(p_6 c_2 s_3 - p_5 s_2 c_3)/(I_2 c_2^2)$$
$$- (-p_4 s_3 + p_5 c_2 c_3 + p_6 s_2 s_3)^2 s_2/(I_2 c_2^3) + M_1 s_3 + M_2 c_3 \tag{214k}$$

$$\dot{p}_6 = (p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3)(p_4 s_3 - p_5 c_2 c_3 - p_6 s_2 s_3)/(I_1 c_2^2)$$
$$+ (-p_4 s_3 + p_5 c_2 c_3 + p_6 s_2 s_3)(p_4 c_3 + p_5 c_2 s_3 - p_6 s_2 c_3)/(I_2 c_2^2) + M_3 \tag{214l}$$

It should be noted that in this example the external forces and torques have remained abstract, as symbolized by $F_\alpha$ and $M_\alpha$ for $\alpha = 1, 2, 3$. Before proceeding with Eqs. (214) as Hamilton's equations, one would be expected to write $F_\alpha$ and $M_\alpha$ in terms of $q$, $p$, and $t$ (eliminating any $\dot{q}$ terms). This task generally requires yet another application of Eq. (202).

*g. Summary for the single rigid body.* On the preceding pages six procedures have been applied to obtain the equations of motion of a single rigid body: the Newton-Euler equations, Lagrange's form of D'Alembert's principle, Kane's quasi-coordinate equations, Lagrange's generalized coordinate equations, Lagrange's quasi-coordinate equations, and Hamilton's equations.

The most obvious result is the fact that only three distinct sets of equations emerged from the six methods. The quasi-coordinate procedures of Kane and Lagrange both reduced to the Newton-Euler equations, as presented in Eqs. (169) and (171). Kane's quasi-coordinate approach is much simpler to apply to this problem than is that of Lagrange.

Lagrange's form of D'Alembert's principle and Lagrange's generalized coordinate equations gave the same results as represented by Eq. (183). These equations are both more difficult to obtain and more difficult to solve (literally or numerically) than are the Newton-Euler equations. The digital computer computational efficiency of Lagrange's equations could be improved in this simple case by a literal inversion of the matrix $M$ shown in Eq. (207).

Hamilton's equations required the most labor to assemble, and required a literal matrix inversion that would be difficult to accomplish for a more complex example. However, the result, as represented by Eqs. (214), has a structure that is more attractive for numerical integration than is that of Lagrange's equations, which have time-varying coefficients of the derivatives of highest order in the system. Hamilton's equations are much more heavily laden with trigonometric functions than are the Newton-Euler equations, however (compare Eq. (214) to the set consisting of Eqs. (169) and (171)).

For this basic example, the Newton-Euler approach is generally[9] the best of the six methods considered, by both of the primary criteria: (1) efficiency of numerical integration, and (2) ease of formulation. In addition, this approach provides equations that are more readily solved literally in certain special cases, such as for $\mathbf{M} = 0$, and most analysts would argue that it offers an advantage in permitting easier physical interpretation of results, particularly when compared with Hamilton's equations.

One must of course be very cautious about generalizing too quickly from a single rigid body to an arbitrary nonrigid spacecraft.

**2. Rigid body with simple nonholonomic constraints.** For systems with holonomic constraints (see Eq. (9)), as for systems with simple nonholonomic constraints, one can record constraint equations in the form $A\dot{q} + B = 0$ (see Eq. (55)).

[9]External force $\mathbf{F}$ and moment $\mathbf{M}$ are unspecified in this example, and the body is of arbitrary shape. For the special case of the axisymmetric frictionless top, simplifications in Lagrange's equations and Hamilton's equations are more dramatic than those occuring in the Newton-Euler equations, and the advantage shifts from vectorial mechanics to analytical mechanics. In Ref. 38, Euler's equations of motion for the top appear as Eq. (9.34), and Lagrange's simpler equations appear as Eq. (9.46).

In spacecraft simulation, one frequently encounters holonomic constraints (such as those relating the four Euler parameters or the nine direction cosines). Nonholonomic constraints, when encountered, are very often in the form of inequalities (such as "stops" on gimbals), and then none of the methods of analytical mechanics is directly applicable (and even a Newton-Euler formulation is awkward). Systems with simple nonholonomic constraints are rather rare in spacecraft attitude simulation work, although they can of course arise in principle, and we should be prepared to deal with them.

In what follows we will examine the classical nonholonomic problem of the homogeneous sphere rolling without slip on a rough surface. We can apply to this problem any of the following methods:

(1) Newton-Euler equations.

(2) Lagrange's form of D'Alembert's principle for simply constrained systems (Subsection II-A-3).

(3) Kane's quasi-coordinate formulation (Subsection II-A-4).

(4) Lagrange's equations for simply constrained systems (Subsection II-B-2).

(5) Lagrange's quasi-coordinate equations (Subsection II-B-3).

(6) Hamilton's equations for simply constrained systems (Subsection II-C-2).

Each of these approaches except the second will be applied here in detail to the rolling sphere problem; the second method is ignored here because it is a special case of the third, and offers no advantages.

*a. Constraint equations.* Common to all methods is the requirement that the sphere rolls without slip on the horizontal plane; thus the point $p$ of the sphere instantaneously in contact with the floor has no velocity relative to the floor, which is assumed to establish an inertial reference frame. (See Fig. 5.) We can incorporate this constraint in the mathematical statement that the velocity of the center $c$ of the sphere is given by

$$\dot{\mathbf{R}} = \boldsymbol{\omega} \times a\mathbf{i}_3 \tag{215}$$

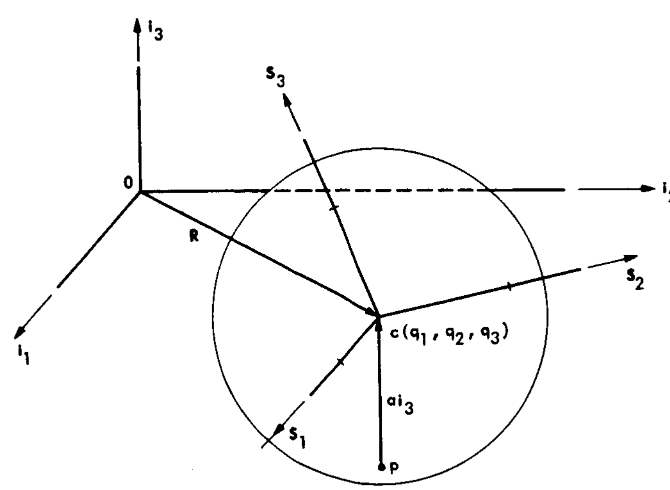


Fig. 5. Sphere rolling without slip

where $\mathbf{R}$ is the position vector of $c$ relative to an inertially fixed point 0, $\boldsymbol{\omega}$ is the inertial angular velocity of $s$, $a$ is the radius of $s$, and $\mathbf{i}_3$ is an inertially fixed unit vector pointing vertically upward.

For all of the formulations to follow, we adopt the set of generalized coordinates $q_1, \cdots, q_6$ such that

$$\mathbf{R} = q_1\mathbf{i}_1 + q_2\mathbf{i}_2 + q_3\mathbf{i}_3 \tag{216a}$$

and, by the definitions illustrated in Fig. 6 for the angles $q_4, q_5, q_6$,

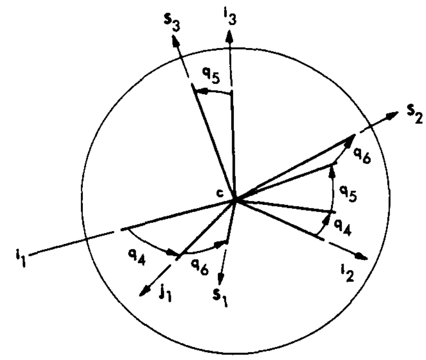


**Fig. 6. Attitude angles for the rolling sphere**

$$\begin{aligned}\boldsymbol{\omega} &= \dot{q}_4\mathbf{i}_3 + \dot{q}_5\mathbf{j}_1 + \dot{q}_6\mathbf{s}_3 \\ &= \mathbf{i}_1\left[\dot{q}_6 \sin q_5 \sin q_4 + \dot{q}_5 \cos q_4\right] \\ &\quad + \mathbf{i}_2\left[\dot{q}_5 \sin q_4 - \dot{q}_6 \sin q_5 \cos q_4\right] \\ &\quad + \mathbf{i}_3\left[\dot{q}_4 + \dot{q}_6 \cos q_5\right]\end{aligned} \tag{216b}$$

In terms of scalars $\omega_1, \omega_2, \omega_3$ defined by

$$\boldsymbol{\omega} = \omega_1\mathbf{i}_1 + \omega_2\mathbf{i}_2 + \omega_3\mathbf{i}_3 \tag{217}$$

Eq. (216b) becomes

$$\begin{aligned}\omega_1 &= \dot{q}_5 \cos q_4 + \dot{q}_6 \sin q_4 \sin q_5 \\ \omega_2 &= \dot{q}_5 \sin q_4 - \dot{q}_6 \cos q_4 \sin q_5 \\ \omega_3 &= \dot{q}_4 + \dot{q}_6 \cos q_5\end{aligned} \tag{218a}$$

and in $3 \times 3$ partitioned matrix terms these kinematical relationships may be expressed as

$$[0 \mid P]\,\dot{q} = \omega \tag{218b}$$

where

$$P \triangleq \begin{bmatrix} 0 & c_4 & s_4 s_5 \\ 0 & s_4 & -c_4 s_5 \\ 1 & 0 & c_5 \end{bmatrix} \tag{218c}$$

with $s_\alpha \triangleq \sin q_\alpha$ and $c_\alpha \triangleq \cos q_\alpha$ for $\alpha = 4, 5$.

Substitution of Eq. (216) into Eq. (215) provides the scalar constraint equations

$$\begin{aligned}\dot{q}_1 - a(\dot{q}_5 \sin q_4 - \dot{q}_6 \cos q_4 \sin q_5) &= 0\\ \dot{q}_2 + a(\dot{q}_5 \cos q_4 + \dot{q}_6 \sin q_4 \sin q_5) &= 0\\ \dot{q}_3 &= 0\end{aligned} \tag{219a}$$

or the matrix equation

$$A\dot{q} = 0 \tag{219b}$$

where in terms of $3 \times 3$ partitions

$$A \triangleq [U \mid A']$$

and

$$A' \triangleq \begin{bmatrix} 0 & -as_4 & ac_4s_5 \\ 0 & ac_4 & as_4s_5 \\ 0 & 0 & 0 \end{bmatrix} \tag{219c}$$

***b. Newton-Euler equations.*** **Direct application of $\mathbf{F} = \mathcal{M}\ddot{\mathbf{R}}$ and $\mathbf{M} = \dot{\mathbf{H}}$ for the sphere leads immediately to**

$$\begin{aligned} F_1 &= \mathcal{M}\ddot{q}_1\\ F_2 &= \mathcal{M}\ddot{q}_2\\ F_3 &= \mathcal{M}\ddot{q}_3\\ M_1 &= I_s\dot{\omega}_1\\ M_2 &= I_s\dot{\omega}_2\\ M_3 &= I_s\dot{\omega}_3 \end{aligned} \tag{220}$$

**where $\mathcal{M}$ is the mass of the sphere, $I_s = \frac{2}{5}\mathcal{M}a^2$ and the scalar components of force $\mathbf{F}$ and moment $\mathbf{M}$ about $c$ are defined by**

$$\mathbf{F} = F_1\mathbf{i}_1 + F_2\mathbf{i}_2 + F_3\mathbf{i}_3 \tag{221a}$$

and

$$\mathbf{M} = M_1\mathbf{i}_1 + M_2\mathbf{i}_2 + M_3\mathbf{i}_3 \tag{221b}$$

Because the idealized rough surface applies a contact force to the sphere at point $p$ only, and there is no other force applied except that due to gravity, we can substitute into Eq. (220) the relationship

$$\begin{aligned} \mathbf{M} &= -a\mathbf{i}_3 \times \mathbf{F}\\ M_1 &= aF_2\\ M_2 &= -aF_1\\ M_3 &= 0 \end{aligned} \tag{222a}$$

or the matrix equation

$$M = -a\tilde{U}^{(3)}F \tag{222b}$$

where

$$\tilde{U}^{(3)} \triangleq \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{222c}$$

Equations (218) through (220) then constitute a complete set of equations for the system.

To facilitate comparison with other formulations, we can record a single matrix equation that embodies all of the system equations. With the definitions

$$\begin{aligned} u_1 &\triangleq \dot{q}_1, & u_2 &\triangleq \dot{q}_2, & u_3 &\triangleq \dot{q}_3 \\ u_4 &\triangleq \omega_1, & u_5 &\triangleq \omega_2, & u_6 &\triangleq \omega_3 \end{aligned} \tag{223}$$

this equation may be written as follows:

$$\left[\begin{array}{c|c|c|c|c} U & 0 & 0 & 0 & 0 \\ \hline 0 & P & 0 & 0 & 0 \\ \hline 0 & 0 & \mathcal{M}U & 0 & -U \\ \hline 0 & 0 & 0 & I_s U & -a\tilde{U}^{(3)} \\ \hline U & A' & 0 & 0 & 0 \end{array}\right] \left\{\begin{array}{c} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \hline \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \\ \hline \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \hline \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \\ \hline F_1 \\ F_2 \\ F_3 \end{array}\right\} = \left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ \hline u_4 \\ u_5 \\ u_6 \\ \hline 0 \\ \hline 0 \\ \hline 0 \end{array}\right\} \tag{224}$$

*c. Lagrange's form of D'Alembert's principle.* Lagrange's form of D'Alembert's principle has been presented here (in Subsection II-A) as a special case of Kane's method, and only the latter has been developed here into a form readily applicable to rigid bodies. Thus we consider for this example only the more general method.

*d. Kane's quasi-coordinate equations.* Equations (55), (64c), and (66b) comprise the system equations proposed by Kane, as developed in Subsection II-A-4. Recall that the first step in this procedure is to use the constraint equations (Eqs. (219)) to solve for the $m$ redundant generalized velocities in terms of the remaining $n$ generalized velocities and the full set of $\nu = n + m$ generalized coordinates. For this example, $m = n = 3$, and Eq. (219) is most easily solved for $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$ in terms of the remaining variables, to obtain

$$\dot{q}_1 = a(\dot{q}_5 \sin q_4 - \dot{q}_6 \cos q_4 \sin q_5) \tag{225a}$$

$$\dot{q}_2 = -a(\dot{q}_5 \cos q_4 + \dot{q}_6 \sin q_4 \sin q_5) \tag{225b}$$

$$\dot{q}_3 = 0 \tag{225c}$$

The next step for the single rigid body is the substitution of Eqs. (225) into expressions for $\dot{\mathbf{R}}$ and $\boldsymbol{\omega}$, using Eqs. (216). At this point in the formulation Kane departs from the pattern established by Lagrange, and selects a set of $n$ quasi-coordinate derivatives $u_1$, $u_2$, and $u_3$ in terms of which $\dot{\mathbf{R}}$ and $\boldsymbol{\omega}$ are easily recorded. For the rolling sphere example the obvious choice is (from Eq. (218a))

$$u_1 \triangleq \omega_1 = \dot{q}_5 \cos q_4 + \dot{q}_6 \sin q_4 \sin q_5 \tag{226a}$$

$$u_2 \triangleq \omega_2 = \dot{q}_5 \sin q_4 - \dot{q}_6 \cos q_4 \sin q_5 \tag{226b}$$

$$u_3 \triangleq \omega_3 = \dot{q}_4 + \dot{q}_6 \cos q_5 \tag{226c}$$

This selection permits $\dot{\mathbf{R}}$ and $\boldsymbol{\omega}$ to be written as simply

$$\dot{\mathbf{R}} = a u_2 \mathbf{i}_1 - a u_1 \mathbf{i}_2 \tag{227a}$$

and

$$\boldsymbol{\omega} = u_1 \mathbf{i}_1 + u_2 \mathbf{i}_2 + u_3 \mathbf{i}_3 \tag{227b}$$

Kane's equations of motion are simply (from Eq. (66a))

$$f_k + f_k^* = 0, \qquad k = 1, \cdots, n \tag{228}$$

where (from Eqs. (65c) and (65d)) for this problem

$$f_k \triangleq \mathbf{F} \cdot \mathbf{V}_k + \mathbf{M} \cdot \boldsymbol{\omega}_k, \qquad k = 1, 2, 3 \tag{229a}$$

$$f_k^* \triangleq -\mathcal{M}\ddot{\mathbf{R}} \cdot \mathbf{V}_k - \dot{\mathbf{H}} \cdot \boldsymbol{\omega}_k, \qquad k = 1, 2, 3 \tag{229b}$$

with $\mathbf{V}_k$ and $\boldsymbol{\omega}_k$ available from Eqs. (65e, 65f, 227a, and 227b) as

$$\begin{aligned} &\mathbf{V}_1 = -a\mathbf{i}_2, \qquad &&\mathbf{V}_2 = a\mathbf{i}_1, \qquad &&\mathbf{V}_3 = 0 \\ &\boldsymbol{\omega}_1 = \mathbf{i}_1, &&\boldsymbol{\omega}_2 = \mathbf{i}_2, &&\boldsymbol{\omega}_3 = \mathbf{i}_3 \end{aligned} \tag{230}$$

Thus we have (with the help of Eqs. (222a) and (227))

$$\begin{aligned} f_1 &= -aF_2 + M_1 = 0 \\ f_2 &= aF_1 + M_2 = 0 \\ f_3 &= 0 \\ f_1^* &= \mathcal{M}a\ddot{R}_2 - I_s\dot{\omega}_1 = -\mathcal{M}a^2\dot{u}_1 - I_s\dot{u}_1 = -(\mathcal{M}a^2 + I_s)\,\dot{u}_1 \\ f_2^* &= -\mathcal{M}a\ddot{R}_1 - I_s\dot{\omega}_2 = -\mathcal{M}a^2\dot{u}_2 - I_s\dot{u}_2 = -(\mathcal{M}a^2 + I_s)\,\dot{u}_2 \\ f_3^* &= -I_s\dot{\omega}_3 = -I_s\dot{u}_3 \end{aligned}$$

The multiplier $\mathcal{M}a^2 + I_s$ is nonzero, so the equations of motion reduce trivially to

$$\begin{aligned} \dot{u}_1 &= 0 \\ \dot{u}_2 &= 0 \\ \dot{u}_3 &= 0 \end{aligned} \tag{231}$$

This simple result is not surprising, although it is seldom realized in the bowling alley. The same implication is available from the Newton-Euler equations appearing in Eq. (224), after a little manipulation. To compare the results of Kane's approach to the Newton-Euler formulation, we can assemble Eqs. (225), (226), and (231) as a single matrix equation equivalent to Eq. (224), as follows:

$$\left[\begin{array}{c|c|c} U & A' & 0 \\ \hline 0 & P & 0 \\ \hline 0 & 0 & U \end{array}\right] \left\{\begin{array}{c} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \hline \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \\ \hline \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline u_1 \\ u_2 \\ u_3 \\ \hline 0 \end{array}\right\} \tag{232}$$

*e. Lagrange's generalized coordinate equations.* As shown in Subsection II-B-2, Eq. (130), Lagrange's generalized coordinate equations for systems with simple constraints appear as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k - \sum_{s=1}^{m} \lambda_s A_{sk} \qquad k = 1, \cdots, \nu \tag{233a}$$

These dynamical equations must be combined with the constraint equations

$$\sum_{k=1}^{\nu} A_{sk}\dot{q}_k + B_s = 0 \qquad s = 1, \cdots, m \tag{233b}$$

to obtain a complete set.

For the rolling sphere example, $\nu = 6$ and $m = 3$. The constraint equations are given by Eq. (219), and the generalized forces $Q_k$ are zero for $k = 1, \cdots, 6$. The kinetic energy of the rigid sphere is given by

$$\begin{aligned} T &= \frac{1}{2}\mathcal{M}\dot{\mathbf{R}}\cdot\dot{\mathbf{R}} + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega} \\ &= \frac{1}{2}\mathcal{M}(\dot{R}_1^2 + \dot{R}_2^2 + \dot{R}_3^2) + \frac{1}{2}I_s(\omega_1^2 + \omega_2^2 + \omega_3^2) \end{aligned} \tag{234}$$

where $I_s = \frac{2}{5}\mathcal{M}a^2$ and $\omega_1, \omega_2, \omega_3$ are defined by Eq. (217). Kinetic energy can be expressed in terms of generalized coordinates and generalized velocities by substituting Eqs. (218a) and (216a) into Eq. (234); the straightforward result is

$$\begin{aligned} T = \frac{1}{2}\mathcal{M}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) + \frac{1}{2}I_s\,[(\dot{q}_5\cos q_4 + \dot{q}_6\sin q_4\sin q_5)^2 \\ + (\dot{q}_5\sin q_4 - \dot{q}_6\cos q_4\sin q_5)^2 + (\dot{q}_4 + \dot{q}_6\cos q_5)^2] \end{aligned} \tag{235a}$$

Before plunging into Eq. (233a) with $T$ in this form, it behooves us to recognize that $T$ can be manipulated and simplified by trigonometric identities into the form

$$T = \frac{1}{2}\mathcal{M}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) + \frac{1}{2}I_s\,[(\dot{q}_4^2 + \dot{q}_5^2 + \dot{q}_6^2 + 2\dot{q}_4\dot{q}_6\cos q_5)] \tag{235b}$$

By substituting from Eqs. (235b) and (219) into Eq. (233a), we can generate the following equations of motion:

$$\mathcal{M}\ddot{q}_1 = -\lambda_1 \tag{236a}$$

$$\mathcal{M}\ddot{q}_2 = -\lambda_2 \tag{236b}$$

$$\mathcal{M}\ddot{q}_3 = -\lambda_3 \tag{236c}$$

$$I_s \frac{d}{dt}(\dot{q}_4 + \dot{q}_6 \cos q_5) = 0 \tag{236d}$$

$$I_s(\ddot{q}_5 + \dot{q}_4\dot{q}_6 \sin q_5) = \lambda_1 a \sin q_4 - \lambda_2 a \cos q_4 \tag{236e}$$

$$I_s \frac{d}{dt}(\dot{q}_6 + \dot{q}_4 \cos q_5) = -\lambda_1 a \cos q_4 \sin q_5 - \lambda_2 a \sin q_4 \sin q_5 \tag{236f}$$

**Equations (236) and (219) comprise a complete set. For comparison with the equivalent Eqs. (224) and (232), it is convenient to introduce the variables $u_1, \cdots, u_6$ defined by**

$$u_j \triangleq \dot{q}_j, \qquad j = 1, \cdots, 6 \tag{237}$$

**and then to write Eqs. (236) and (219) as a single first order matrix differential equation. After expanding the derivatives in Eqs. (236d) and (236f) and substituting the new variables, we find the following system equation:**

$$\left[\begin{array}{c|c|c|c|c} U & 0 & 0 & 0 & 0 \\ \hline 0 & U & 0 & 0 & 0 \\ \hline 0 & 0 & \mathcal{M}U & 0 & U \\ \hline 0 & 0 & 0 & I' & A'^T \\ \hline U & A' & 0 & 0 & 0 \end{array}\right] \left(\begin{array}{c} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \hline \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \\ \hline \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \hline \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \\ \hline \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{array}\right) = \left(\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ \hline u_4 \\ u_5 \\ u_6 \\ \hline 0 \\ \hline I_s \sin q_5 \begin{Bmatrix} u_5 u_6 \\ -u_4 u_6 \\ u_4 u_5 \end{Bmatrix} \\ \hline 0 \end{array}\right) \tag{238}$$

where

$$I' \triangleq I_s \begin{bmatrix} 1 & 0 & \cos q_5 \\ 0 & 1 & 0 \\ \cos q_5 & 0 & 1 \end{bmatrix}$$

*f. Lagrange's quasi-coordinate equations.* Equations of motion for the rolling sphere could also be written from Eq. (147a), using the Lagrangian quasi-coordinate formulation. The results will of course depend on the definitions adopted for the elements of $u$. As noted in Subsection II-B-3 in general terms, the choice indicated by Eq. (237) produces the result just obtained as Eq. (238), and offers no advantage over the generalized coordinate method of Lagrange. The alternative natural choice is that displayed in Eq. (223); this offers the advantage of producing $\overline{T}$ from Eq. (234) in the simple form

$$\overline{T} = \frac{1}{2}\mathcal{M}(u_1^2 + u_2^2 + u_3^2) + \frac{1}{2}I_s(u_4^2 + u_5^2 + u_6^2) \tag{239}$$

With this choice for $u$, we have $\omega = 0$ and

$$W \triangleq \left[\begin{array}{c|c} U & 0 \\ \hline 0 & P^T \end{array}\right] \tag{240}$$

with $P$ as given by Eq. (218c).

With the restrictions applicable to the rolling sphere problem with $u$ defined by Eq. (223), Eq. (148) reduces to

$$W\frac{d}{dt}(\overline{T}_{,u}) + [u^T W^{-1} W_{ij,q}]\overline{T}_{,u} - \{u^T W^{-1} W_{,q_k}\overline{T}_{,u}\} + A^T\lambda = 0 \tag{241}$$

The basic new ingredients of this equation are given by

$$\overline{T}_{,u} = \{\mathcal{M}u_1 \quad \mathcal{M}u_2 \quad \mathcal{M}u_3 \quad I_s u_4 \quad I_s u_5 \quad I_s u_6\}^T \tag{242}$$

$$W^{-1} = \left[\begin{array}{c|c} U & 0 \\ \hline 0 & (P^T)^{-1} \end{array}\right], \quad \text{where } (P^T)^{-1} = -\frac{1}{s_5}\begin{bmatrix} s_4c_5 & -c_4s_5 & -s_4 \\ -c_4c_5 & -s_4s_5 & c_4 \\ -s_5 & 0 & 0 \end{bmatrix} \tag{243}$$

$$W_{,q_k} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & P^T_{,q_k} \end{array}\right], \quad \text{where for } k = 1, 2, 3, \text{ and } 6 \quad P^T_{,q_k} = 0 \tag{244a}$$

and

$$P^T_{,q_4} = \begin{bmatrix} 0 & 0 & 0 \\ -s_4 & c_4 & 0 \\ c_4s_5 & s_4s_5 & 0 \end{bmatrix} \tag{244b}$$

$$P^T_{,q_5} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ s_4c_5 & -c_4c_5 & -s_5 \end{bmatrix} \tag{244c}$$

and finally by

$$W_{ij,q} = 0, \qquad \text{for } i = 1, 2, 3, 4 \text{ and for } j = 1, 2, 3 \tag{245a}$$

and

$$W_{54,q} = \{0 \quad 0 \quad 0 \quad -s_4 \quad 0 \quad 0\}^T \tag{245b}$$

$$W_{55,q} = \{0 \quad 0 \quad 0 \quad c_4 \quad 0 \quad 0 \quad \}^T \tag{245c}$$

$$W_{56,q} = \{0\}$$

$$W_{64,q} = \{0 \quad 0 \quad 0 \quad c_4s_5 \quad s_4c_5 \quad 0\}^T \tag{245d}$$

$$W_{65,q} = \{0 \quad 0 \quad 0 \quad s_4s_5 \quad -c_4c_5 \quad 0\}^T \tag{245e}$$

$$W_{66,q} = \{0 \quad 0 \quad 0 \quad 0 \quad -s_5 \quad 0\}^T \tag{245f}$$

The matrix product $u^TW^{-1}$, appearing twice in Eq. (241), is given by

$$u^TW^{-1} = \{u_1 \quad u_2 \quad u_3 \quad (-s_4c_5u_4 + c_4c_5u_5 + s_5u_6)/s_5 \quad (c_4u_4 + s_4u_5) \quad (s_4u_4 - c_4u_5)/s_5\} \tag{246}$$

The $6 \times 6$ matrix $[u^TW^{-1}W_{ij,q}]$ is null except for the following entries:

$$u^TW^{-1}W_{54,q} = (s_4^2c_5u_4 - s_4c_4c_5u_5 - s_4s_5u_6)/s_5 \tag{247a}$$

$$u^TW^{-1}W_{55,q} = (-s_4c_4c_5u_4 + c_4^2c_5u_5 + c_4s_5u_6)/s_5 \tag{247b}$$

$$u^TW^{-1}W_{64,q} = -s_4c_4c_5u_4 + c_4^2c_5u_5 + c_4s_5u_6 + s_4c_4c_5u_4 + s_4^2c_5u_5 = c_5u_5 + c_4s_5u_6 \tag{247c}$$

$$\begin{aligned} u^TW^{-1}W_{65,q} &= -s_4^2c_5u_4 + s_4c_4c_5u_5 + s_4s_5u_6 - c_4^2c_5u_4 - s_4c_4c_5u_5 \\ &= -c_5u_4 + s_4s_5u_6 \end{aligned} \tag{247d}$$

$$u^TW^{-1}W_{66,q} = -c_4s_5u_4 - s_4s_5u_5 \tag{247e}$$

The symbol combination $u^TW^{-1}W_{,q_k}$ in Eq. (241) describes the $k$th row of a $6 \times 6$ matrix. Examination of Eq. (244) reveals that these rows are filled with zeros for $k = 1, 2, 3$, and 6, while the combination of Eqs. (244) and (246) provides

$$\begin{aligned} u^TW^{-1}W_{,q_4} &= \{0 \quad 0 \quad 0 \quad (-s_4c_4u_4 - s_4^2u_5 + s_4c_4u_4 - c_4^2u_5) \quad (c_4^2u_4 + s_4c_4u_5 + s_4^2u_4 - s_4c_4u_5) \quad 0\} \\ &= \{0 \quad 0 \quad 0 \quad -u_5 \quad u_4 \quad 0\} \end{aligned} \tag{248a}$$

and

$$u^T W^{-1} W_{,q_5} = \{0 \quad 0 \quad 0 \quad (s_4^2 c_5 u_4 - s_4 c_4 c_5 u_5)/s_5 \quad (-s_4 c_4 c_5 u_4 + c_4^2 c_5 u_5)/s_5 \quad (-s_4 u_4 + c_4 u_5)\} \tag{248b}$$

Finally the four terms in Eq. (241) can be considered as individual column matrices to be added together. The first term is obtained from Eqs. (240) and (242) as

$$W \frac{d}{dt}(\overline{T}_{,u}) = \begin{Bmatrix} \mathcal{M}\dot{u}_1 \\ \mathcal{M}\dot{u}_2 \\ \mathcal{M}\dot{u}_3 \\ I_s \dot{u}_6 \\ I_s (c_4 \dot{u}_4 + s_4 \dot{u}_5) \\ I_s (s_4 s_5 \dot{u}_4 - c_4 s_5 \dot{u}_5 + c_5 \dot{u}_6) \end{Bmatrix} \tag{249a}$$

The second term in Eq. (241) is, from Eqs. (242) and (247),

$$[u^T W^{-1} W_{ij,q}]\, \overline{T}_{,u} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_s (s_4^2 c_5 u_4^2 + c_4^2 c_5 u_5^2 - 2 s_4 c_4 c_5 u_4 u_5 - s_4 s_5 u_4 u_6 + c_4 s_5 u_5 u_6)/s_5 \\ 0 \end{Bmatrix} \tag{249b}$$

The third term in Eq. (241) is, from Eqs. (242) and (248),

$$-\{u^T W^{-1} W_{,q_k} \overline{T}_{,u}\} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_s (s_4^2 c_5 u_4^2 + c_4^2 c_5 u_5^2 - 2 s_4 c_4 c_5 u_4 u_5 - s_4 s_5 u_4 u_6 + c_4 s_5 u_5 u_6)/s_5 \\ 0 \end{Bmatrix} \tag{249c}$$

The fourth and final term in Eq. (241) is available from Eq. (219) as

$$A^T \lambda = \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ 0 \\ -a s_4 \lambda_1 + a c_4 \lambda_2 \\ a c_4 s_5 \lambda_1 + a s_4 s_5 \lambda_2 \end{Bmatrix} \tag{249d}$$

Finally we can see that when Eqs. (249) are combined as required by Eq. (241) there emerge the simple scalar equations that follow:

$$\mathcal{M}\dot{u}_1 + \lambda_1 = 0 \tag{250a}$$

$$\mathcal{M}\dot{u}_2 + \lambda_2 = 0 \tag{250b}$$

$$\mathcal{M}\dot{u}_3 + \lambda_3 = 0 \tag{250c}$$

$$I_s\dot{u}_6 = 0 \tag{250d}$$

$$I_s(\dot{u}_4 \cos q_4 + \dot{u}_5 \sin q_4) - \lambda_1 a \sin q_4 + \lambda_2 a \cos q_4 = 0 \tag{250e}$$

$$\begin{aligned} I_s(\dot{u}_4 \sin q_4 \sin q_5 - \dot{u}_5 \cos q_4 \sin q_5 + \dot{u}_6 \cos q_5) \\ + \lambda_1 a \cos q_4 \sin q_5 + \lambda_2 a \sin q_4 \sin q_5 = 0 \end{aligned} \tag{250f}$$

Equations (250) must be combined with the constraint equations (219) and the definitions in Eq. (223) before the set is complete. When the result is cast as a single martix equation for comparison with Eqs. (224), (232), and (238), we have

$$\left[\begin{array}{c|c|c|c|c} U & 0 & 0 & 0 & 0 \\ \hline 0 & P & 0 & 0 & 0 \\ \hline 0 & 0 & \mathcal{M}U & 0 & U \\ \hline 0 & 0 & 0 & I'' & A'^T \\ \hline U & A' & 0 & 0 & 0 \end{array}\right] \left\{\begin{array}{c} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \hline \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \\ \hline \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \hline \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \\ \hline \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{array}\right\} = \left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ \hline u_4 \\ u_5 \\ u_6 \\ \hline 0 \\ \hline 0 \\ \hline 0 \end{array}\right\} \tag{251}$$

where

$$I'' \triangleq I_s \begin{bmatrix} 0 & 0 & 1 \\ \cos q_4 & \sin q_4 & 0 \\ \sin q_4 \sin q_5 & -\cos q_4 \sin q_5 & \cos q_5 \end{bmatrix}$$

For future reference, it may be noted that if we had chosen for this example to write our equation of motion in the compressed form established by Eq. (147b), we would have

$$\frac{d}{dt}(\bar{T}_{,u}) + W^{-1}\gamma\bar{T}_{,u} + W^{-1}A^T\lambda = 0$$

If we had then explicitly written out the coefficient of $\bar{T}_{,u}$, using Eqs. (243), and (246) through (248), we would have found

$$W^{-1}\gamma = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -\tilde{\omega} \end{array}\right] \tag{252}$$

This result differs by a sign from the expression recorded in Eq. (198), and serves as a reminder of the limitations of that equation. The symbols $\omega_1$, $\omega_2$, $\omega_3$ in the example leading to Eq. (198) are scalar components of the inertial angular velocity of the body for a *body-fixed vector basis*, while in the present example an inertially-fixed vector basis is used (see Eq. (217)). The latter choice is reasonable only for a sphere, since any other body would have time-varying inertia properties for the inertially fixed vector basis. For the sphere, as we have seen, the term $W^{-1}\gamma \overline{T}_{,u}$ is simply zero.

*g. Hamilton's equations.* In applying Hamilton's equations in the form of Eq. (164) to the rolling sphere, it is immediately apparent that the resulting system equations will have the same basic dimension and structure as Eqs. (238) and (251), with generalized momenta replacing the quasi-coordinate derivatives (of which they are a special case). The $6 \times 6$ matrix in the upper left corner of the coefficient matrix in Eq. (251) would be replaced by the matrix for this system defined as $M$ in Eq. (201b), and the matrices $\mathcal{M}U$ and $I''$ in Eq. (251) would both become simply $U$. The third and fourth partitions on the right side of Eq. (251) would become respectively $\mathcal{H}_{,p}$ and $\mathcal{H}_{,q}$. The partition equations would be rearranged to establish the symmetry of Eq. (164).

To obtain $\mathcal{H}_{,p}$ and $\mathcal{H}_{,q}$ explicitly it is necessary to invert $M$ literally. This is not an extraordinary task, because the $6 \times 6$ matrix $M$ is diagonal except for the $3 \times 3$ partition in the lower right hand corner, appearing in detail as

$$M = \left[\begin{array}{c|ccc} \mathcal{M}U & & 0 & \\ \hline 0 & I_s & 0 & I_s c_5 \\ & 0 & I_s & 0 \\ & I_s c_5 & 0 & I_s \end{array}\right] \tag{253a}$$

Inversion provides

$$M^{-1} = \left[\begin{array}{c|c} U/\mathcal{M} & 0 \\ \hline 0 & \begin{bmatrix} 1 & 0 & -c_5 \\ 0 & s_5^2 & 0 \\ -c_5 & 0 & 1 \end{bmatrix} \Big/ (I_s s_5^2) \end{array}\right] \tag{253b}$$

so that from Eq. (202) we have

$$\dot{q}_1 = p_1/\mathcal{M} \tag{254a}$$

$$\dot{q}_2 = p_2/\mathcal{M} \tag{254b}$$

$$\dot{q}_3 = p_3/\mathcal{M} \tag{254c}$$

$$\dot{q}_4 = (p_4 - c_5 p_6)/(I_s s_5^2) \tag{254d}$$

$$\dot{q}_5 = p_5/I_s \tag{254e}$$

$$\dot{q}_6 = (p_6 - c_5 p_4)/(I_s s_5^2) \tag{254f}$$

For this problem the Hamiltonian $\mathcal{H}$ is simply the kinetic energy $T$, so Eqs. (235) and (254) can be combined to yield (after some manipulation)

$$\mathcal{H} = (p_1^2 + p_2^2 + p_3^2)/(2\mathcal{M}) + (p_4^2 + p_5^2 s_5^2 + p_6^2 - 2p_4 p_6 c_5)/(2I_s s_5^2) \tag{255}$$

The partial derivatives $\mathcal{H}_{,q}$ and $\mathcal{H}_{,p}$ can be obtained from Eq. (255). The column matrix $\mathcal{H}_{,q}$ is null except for the element in the fifth row, which can be shown to be

$$\mathcal{H}_{,q_5} = [p_4 p_6 (1 + c_5^2) - c_5 (p_4^2 + p_6^2)]/(I_s s_5^3) \tag{256}$$

Partial differentiation of Eq. (255) also provides

$$\mathcal{H}_{,p} = \left\{ \begin{array}{c} p_1/\mathcal{M} \\ p_2/\mathcal{M} \\ p_3/\mathcal{M} \\ (p_4 - c_5 p_6)/(I_s s_5^2) \\ p_5/I_s \\ (p_6 - c_5 p_4)/(I_s s_5^2) \end{array} \right\} \tag{257}$$

Thus the system equations stemming from a Hamiltonian formulation may be arranged as follows, in conformity with the symmetric pattern indicated by Eq. (164):

$$\left[ \begin{array}{c|c|c|c|c} 0 & 0 & U & 0 & U \\ \hline 0 & 0 & 0 & U & A'^T \\ \hline U & 0 & 0 & 0 & 0 \\ \hline 0 & U & 0 & 0 & 0 \\ \hline U & A' & 0 & 0 & 0 \end{array} \right] \left\{ \begin{array}{c} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \hline \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \\ \hline \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \hline \dot{p}_4 \\ \dot{p}_5 \\ \dot{p}_6 \\ \hline \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ -\mathcal{H}_{,q_5} \\ 0 \\ \hline p_1/\mathcal{M} \\ p_2/\mathcal{M} \\ p_3/\mathcal{M} \\ \hline (p_4 - c_5 p_6)/(I_s s_5^2) \\ p_5/I_s \\ (p_6 - c_5 p_4)/(I_s s_5^2) \\ \hline 0 \\ 0 \\ 0 \end{array} \right\} \tag{258}$$

*h. Summary for the rolling sphere.* Equations (224), (232), (238), (251), and (258) all provide complete system equations for the sphere rolling without slip on a horizontal surface, but these matrix equations differ in dimension and in amenability to literal solution and numerical integration.

The number of first order scalar differential equations is fifteen in every case but one; Kane's quasi-coordinate formulation produces only nine such equations in Eq. (232). Moreover, three of these equations (the dynamics equations) are trivially satisfied by constants, and the right side of the remaining equations consists only of these constants. Equation (232) is not only the smallest in dimension, but also the only set that yields an obvious partial solution in closed form. Complete solution would require a numerical integration of only six first order equations.

Although five formulations have been presented, only three distinct sets of kinematic variables and two sets of kinetic variables (constraint forces and equivalent Lagrange multipliers) have been employed. Only the choice of kinematic variables is important here. In Eqs. (224), (232), and (251) the kinematic variables included angular velocity scalar components as well as the generalized coordinates illustrated in Figs. 5 and 6. For the Lagrangian formulation leading to Eq. (251) we were obliged to work with generalized coordinates and generalized velocities, while for the Hamiltonian formulation we had generalized coordinates and generalized momenta to deal with. The simplicity of Eq. (232) suggests that its kinematic unknowns are best suited to this problem. Since Eqs. (224) and (251) employ the same variables as Eq. (232), perhaps with sufficient manipulation they too would yield partial explicit literal solutions. This seems less likely for either the Lagrangian or the Hamiltonian formulation.

If we ignore the possibility of closed-form literal solution, and consider the relative merits of Eqs. (224), (232), (238), (251), and (258) as the subject of a digital computer numerical integration, then our standards of judgment change. Equation (232), based on Kane's method, is still much favored because its dimensions are small. Of the remaining options, the most attractive set is Eq. (258), based on Hamilton's equations. The left side coefficient matrix in Eq. (258) has the unique advantage of symmetry, and contains only one time-varying $3 \times 3$ partition (appearing with its transpose). Equation (224), from the Newton-Euler formulation, has two distinct time-varying partitions in its coefficient matrix, as does the Lagrangian set in Eq. (238). The Lagrangian quasi-coordinate equations (Eq. (251)) have a simpler right side than the generalized coordinate equations, but only at the expense of complicating the more important left side with four time-varying coefficient matrix partitions, three of which are distinct.

Before we accept too sweepingly the implied ranking of these results in order of diminishing merit (Eqs. 232, 258, 224, 238, and finally 251), we should pause to examine the labors of derivation and the difficulties of extension to systems of higher dimension. It is significant that the two methods yielding the best results both involve a matrix inversion that must be performed literally (nonnumerically) even for problems of higher dimension. For this problem the "matrix inversion" involved in Kane's method was trivial; in effect we inverted a unit matrix in recording Eq. (225) from Eq. (219). Indeed the method is sufficiently flexible to permit the analyst to choose his preferred variables so as to minimize the labors of inversion. But with the Hamiltonian formulation all flexibility for the analyst is sacrificed; he must either literally invert the matrix $M$ defined in Eq. (201b), so he can obtain literal expressions for the partial derivatives in $\mathcal{H}_{,p}$ and $\mathcal{H}_{,q}$, or he must accept the computational burdens of the identity

$$(M^{-1})_{,q_\alpha} = -M^{-1}M_{,q_\alpha}M^{-1}$$

**3. Symmetric three-body system with small deformations.** Figure 7 portrays a system of three point-connected rigid bodies with linearly elastic hinges that generate interbody torques when the system departs from the nominal configuration represented by dashed lines. There are no external forces or torques applied. Attention is focused on the free vibration problem, for which the deformation angles $\gamma_1$ and $\gamma_2$ and their time derivatives remain "small," while the central body experiences arbitrary rotations in inertial space. The term "small" formally means *arbitrarily small*; only linear terms in the kinematic variables of deformation are retained in the final equations of motion.

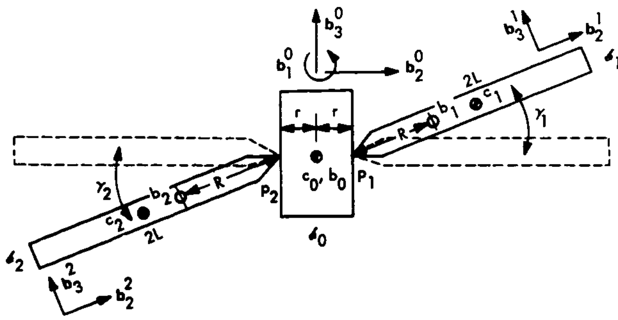

Fig. 7. Symmetric three-body system

Although virtually all of the methods discussed in this report could be applied to this system, we can draw upon our experience with previous examples to anticipate that several methods will give the same results, and that other methods will demand unreasonable labors of derivation in return for system equations of relatively greater complexity. Accordingly, equations of motion are derived here in detail only for Kane's method and for the Lagrangian quasi-coordinate method; the Newton-Euler equations are recorded from another reference, and the advantages and disadvantages of the Lagrangian generalized coordinate approach and Hamilton's formulation are discussed without generating a complete set of equations.

**a. *Newton-Euler equations.*** In Appendix C of Ref. 31 the three-body example appearing in Fig. 7 is used to illustrate an algorithm for digital computer formulation of equations of motion of $n + 1$ rigid bodies that are interconnected by $n$ line hinges, and hence in a topological tree configuration. Eq. (1) of Ref. 31 is an explicit representation of the unrestricted equations of motion of such a system; partial linearization of this generic equation produces for this three-body example the following equations of motion:

$$\begin{aligned} I_1\dot{\omega}_1 + (J + \mathcal{M}Rr)(\ddot{\gamma}_1 + \ddot{\gamma}_2) = {} & (I_2 - I_3)\,\omega_2\omega_3 \\ & + (J + \mathcal{M}Rr)(\omega_2^2 - \omega_3^2)(\gamma_1 + \gamma_2) \end{aligned} \tag{259a}$$

$$\begin{aligned} I_2\dot{\omega}_2 - (J + \mathcal{M}Rr)(\gamma_1 + \gamma_2)\,\dot{\omega}_3 = {} & (I_3 - I_1)\,\omega_3\omega_1 \\ & - (J + \mathcal{M}Rr)\,\omega_1\omega_2(\gamma_1 + \gamma_2) \end{aligned} \tag{259b}$$

$$\begin{aligned} I_3\dot{\omega}_3 - (J + \mathcal{M}Rr)(\gamma_1 + \gamma_2)\,\dot{\omega}_2 = {} & (I_1 - I_2)\,\omega_1\omega_2 \\ & + 2(J + \mathcal{M}Rr)\,\omega_2(\dot{\gamma}_1 + \dot{\gamma}_2) \\ & + (J + \mathcal{M}Rr)\,\omega_1\omega_3(\gamma_1 + \gamma_2) \end{aligned} \tag{259c}$$

$$\begin{aligned} (J + \mathcal{M}Rr)\,\dot{\omega}_1 + (J - \mathcal{M}R^2)\ddot{\gamma}_1 + \mathcal{M}R^2\ddot{\gamma}_2 = {} & -(J + \mathcal{M}Rr)\,\omega_2\omega_3 \\ & + [J(\omega_2^2 - \omega_3^2) - \mathcal{M}R^2(\omega_1^2 + \omega_2^2) \\ & - \mathcal{M}Rr(\omega_1^2 + \omega_3^2)]\,\gamma_1 \\ & + \mathcal{M}R^2(\omega_1^2 + \omega_2^2)\,\gamma_2 - k\gamma_1 \end{aligned} \tag{259d}$$

$$(J + \mathcal{M}Rr)\dot{\omega}_1 + \mathcal{M}R^2\ddot{\gamma}_1 + (J - \mathcal{M}R^2)\ddot{\gamma}_2 = -(J + \mathcal{M}Rr)\,\omega_2\omega_3 + [J(\omega_2^2 - \omega_3^2) - \mathcal{M}R^2(\omega_1^2 + \omega_2^2) - \mathcal{M}Rr(\omega_1^2 + \omega_3^2)]\,\gamma_2 + \mathcal{M}R^2(\omega_1^2 + \omega_2^2)\,\gamma_1 - k\gamma_2 \tag{259e}$$

Here $I_1, I_2, I_3$ are the principal moments of inertia of the three-body system in its nominal configuration (with $\gamma_1 = \gamma_2 = 0$), referred to the system mass center. The inertial angular velocity $\boldsymbol{\omega}$ of the central body $\mathscr{b}_0$ defines the scalars $\omega_j \stackrel{\Delta}{=} \boldsymbol{\omega} \cdot \mathbf{b}_j^0$ $(j = 1,2,3)$, in terms of unit vectors $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$ shown in Fig. 7. The angles $\gamma_1$ and $\gamma_2$ are shown as positive angles in Fig. 7. The appendages $\mathscr{b}_1$ and $\mathscr{b}_2$ in the figure are each uniform thin rods of mass per unit length $\mu$, mass $m$, and length $2L$ (so that $m \stackrel{\Delta}{=} 2\mu L$); the central body $\mathscr{b}_0$ is a homogeneous cylinder of radius $r$, and the total system mass is $\mathcal{M}$. The symbols $R$ and $J$ in Eqs. (259) are defined by

$$R \stackrel{\Delta}{=} mL/\mathcal{M} \tag{260a}$$

$$J \stackrel{\Delta}{=} 4mL^2/3 \tag{260b}$$

the latter being the moment of inertia of a single rod about its hinge axis, which is parallel to $\mathbf{b}_1^0$. Rotations of appendages relative to the main body are resisted by linear rotary springs with identical spring constants $k$.

The origin of Eq. (1) of Ref. 31 is in the Newton-Euler equations for each of the $n + 1$ bodies of the system, and in this indirect sense Eqs. (259) are derived in Ref. 31 by a Newton-Euler formulation. As noted in that reference, however, it is of course also possible (and more efficient for an experienced analyst without a ready-made computer program) to obtain these equations from the first principles of Newton and Euler. Equations (259a) through (259c) can be obtained simply by taking the inertial frame time derivative of the system angular momentum for the system mass center. Equation (259d) is available from the dot product of $\mathbf{b}_1^0$ and the equation (see Ref. 38, p. 410)

$$\mathbf{M}^{p_1} = \dot{\mathbf{H}}^{p_1} + m\mathbf{c}_1 \times \ddot{\mathbf{P}}_1 \tag{261}$$

where $\mathbf{M}^{p_1}$ is the moment about $p_1$ applied to $\mathscr{b}_1$, $\mathbf{H}^{p_1}$ is the inertial angular momentum of $\mathscr{b}_1$ referred to $p_1$, $\mathbf{c}_1 \stackrel{\Delta}{=} L\mathbf{b}_2^1$, and $\mathbf{P}_1$ is the vector from the (inertially stationary) system mass center to $p_1$. Equation (259e) is available from a similar calculation, applied to $\mathscr{b}_2$.

*After* Eqs. (259) are obtained by a Newton-Euler formulation, it is possible (by inspection in this case and by eigenvalue analysis in the general case of nominally constant $\boldsymbol{\omega}$) to discover a coordinate transformation that simplifies the appearance of the equations of motion. In this example it is clear that Eqs. (259) will be simpler if rewritten in terms of the new generalized coordinates

$$\eta_1 = \gamma_1 + \gamma_2 \tag{262a}$$

$$\eta_2 = \gamma_1 - \gamma_2 \tag{262b}$$

Equations (259a through c) can be rewritten directly in terms of the new coordinates, and Eqs. (259d, e) can be replaced by their sum and their difference and then transformed. The result is

$$I_1\dot{\omega}_1 + (J + \mathcal{M}Rr)\ddot{\eta}_1 = (I_2 - I_3)\,\omega_2\omega_3 + (J + \mathcal{M}Rr)(\omega_2^2 - \omega_3^2)\,\eta_1 \tag{263a}$$

$$I_2\dot{\omega}_2 - (J + \mathcal{M}Rr)\,\dot{\omega}_3\eta_1 = (I_3 - I_1)\,\omega_3\omega_1 - (J + \mathcal{M}Rr)\,\omega_1\omega_2\eta_1 \tag{263b}$$

$$I_3\dot{\omega}_3 - (J + \mathcal{M}Rr)\,\dot{\omega}_2\eta_1 = (I_1 - I_2)\,\omega_1\omega_2 + 2(J + \mathcal{M}Rr)\,\omega_2\dot{\eta}_1 + (J + \mathcal{M}Rr)\,\omega_1\omega_3\eta_1 \tag{263c}$$

$$J\ddot{\eta}_1 + 2(J + \mathcal{M}Rr)\,\dot{\omega}_1 = -2(J + \mathcal{M}Rr)\,\omega_2\omega_3 + [J(\omega_2^2 - \omega_3^2) - \mathcal{M}Rr\,(\omega_1^2 + \omega_3^2) - k]\,\eta_1 \tag{263d}$$

$$(J - 2\mathcal{M}R^2)\,\ddot{\eta}_2 = [J(\omega_2^2 - \omega_3^2) - 2\mathcal{M}R^2(\omega_1^2 + \omega_2^2) - \mathcal{M}Rr\,(\omega_1^2 + \omega_3^2)]\,\eta_2 \tag{263e}$$

Because only the final equation involves $\eta_2$, and this equation is satisfied by $\eta_2 \equiv 0$, it is quite satisfactory for the purposes of dynamic analysis of rotational motions of the central body to truncate this system of equations, abandoning Eq. (263e) entirely. For symmetric systems such as this example, it might be apparent even before the equations are derived that irrelevant generalized coordinates can be identified and equations in these coordinates eliminated from the equations of motion. It is a shortcoming of the Newton-Euler approach (in contrast with the methods of Lagrange and Hamilton) that such coordinates cannot be excluded from the formulation at the outset.

***b. Kane's quasi-coordinate equations.*** As developed in Subsection II-A-4, Kane's quasi-coordinate formulation of D'Alembert's principle can be applied to the three-body example in Fig. 7 by selecting the quasi-coordinate derivatives[10]

$$u_1 \triangleq \dot{\gamma}_1, \qquad u_2 \triangleq \dot{\gamma}_2 \tag{264a}$$

$$u_3 \triangleq \omega_1, \qquad u_4 \triangleq \omega_2, \qquad u_5 \triangleq \omega_3 \tag{264b}$$

and recording the equations of motion (see Eq. 66a)

$$f_k + f_k^* = 0, \qquad k = 1, \cdots, 5 \tag{265}$$

where (as in Eqs. 65c and 65d)

$$f_k \triangleq \sum_{j=0}^{2} [\mathbf{F}^j \cdot \mathbf{V}_k^{c_j} + \mathbf{M}^j \cdot \boldsymbol{\omega}_k^j], \qquad k = 1, \cdots, 5 \tag{266a}$$

and

$$f_k^* \triangleq -\sum_{j=0}^{2} [\mathcal{M}_j \ddot{\mathbf{R}}^j \cdot \mathbf{V}_k^{c_j} + \dot{\mathbf{H}}^j \cdot \boldsymbol{\omega}_k^j], \qquad k = 1, \cdots, 5 \tag{266b}$$

[10]We could alternatively choose $\dot{\eta}_1 \triangleq u_1$ and $\dot{\eta}_2 \triangleq u_2$, noting Eqs. (262) and (263).

The vectors $\mathbf{V}^{c_j}_k$ and $\boldsymbol{\omega}^j_k$ are obtained from

$$\dot{\mathbf{R}}^j \triangleq \sum_{k=1}^{5} \mathbf{V}^{c_j}_k u_k + \mathbf{V}^{c_j}_t \qquad (267a)$$

$$\boldsymbol{\omega}^j \triangleq \sum_{k=1}^{5} \boldsymbol{\omega}^j_k u_k + \boldsymbol{\omega}_t \qquad (267b)$$

where $\dot{\mathbf{R}}^j$ and $\boldsymbol{\omega}^j$ are respectively the mass center inertial velocity and the inertial angular velocity of the $j$th body.

For this example, the inertial angular velocities are

$$\boldsymbol{\omega}^0 = \omega_1 \mathbf{b}^0_1 + \omega_2 \mathbf{b}^0_2 + \omega_3 \mathbf{b}^0_3 = u_3 \mathbf{b}^0_1 + u_4 \mathbf{b}^0_2 + u_5 \mathbf{b}^0_3 \qquad (268a)$$

$$\boldsymbol{\omega}^1 = \boldsymbol{\omega}^0 + \dot{\gamma}_1 \mathbf{b}^0_1 = (u_1 + u_3)\,\mathbf{b}^0_1 + u_4 \mathbf{b}^0_2 + u_5 \mathbf{b}^0_3 \qquad (268b)$$

$$\boldsymbol{\omega}^2 = \boldsymbol{\omega}^0 + \dot{\gamma}_2 \mathbf{b}^0_1 = (u_2 + u_3)\,\mathbf{b}^0_1 + u_4 \mathbf{b}^0_2 + u_5 \mathbf{b}^0_3 \qquad (268c)$$

Since the three-body system is force-free, its mass center is inertially stationary, and we can measure $\mathbf{R}^0$, $\mathbf{R}^1$, and $\mathbf{R}^2$ from that point. Mass center definition then provides the relationship

$$m\,(r\mathbf{b}^0_2 + L\mathbf{b}^1_2) + m\,(-r\mathbf{b}^0_2 - L\mathbf{b}^2_2) + (\mathcal{M} - 2m)\,(0) = -\mathcal{M}\mathbf{R}^0$$

or

$$\mathbf{R}^0 = (\mathbf{b}^2_2 - \mathbf{b}^1_2)\,mL/\mathcal{M} = R\,(\mathbf{b}^2_2 - \mathbf{b}^1_2) \qquad (269a)$$

in which the definition of $R$ in Eq. (260a) has been employed. With the additional position vectors

$$\mathbf{R}^1 = \mathbf{R}^0 + r\mathbf{b}^0_2 + L\mathbf{b}^1_2 = r\mathbf{b}^0_2 + (L - R)\,\mathbf{b}^1_2 + R\mathbf{b}^2_2 \qquad (269b)$$

and

$$\mathbf{R}^2 = \mathbf{R}^0 - r\mathbf{b}^0_2 - L\mathbf{b}^2_2 = -r\mathbf{b}^0_2 - R\mathbf{b}^1_2 - (L - R)\,\mathbf{b}^2_2 \qquad (269c)$$

we can differentiate twice with respect to time and complete the kinematic analysis required by Kane's procedure. Before proceeding, however, we should examine our limited "small deformation" goals; simplifying assumptions should be incorporated as soon as possible for most efficient analysis. In view of our decision to linearize in $\gamma_1, \gamma_2, \dot{\gamma}_1, \dot{\gamma}_2, \ddot{\gamma}_1, \ddot{\gamma}_2$, we might be tempted to try to eliminate much of the labor from our problem by dropping all nonlinear terms in $\gamma_1$ and $\gamma_2$ (and their derivatives) at the outset of the kinematic analysis, prior to differentiation. We must be careful, however, to retain all linear terms in every vector appearing in Eqs. (266), and this implies the retention of certain nonlinear terms in Eqs. (267). (Similar precautions must be taken with Lagrangian and Hamiltonian formulation, since partial differentiations of second degree terms in the kinematic equations can produce first degree terms in the equations of motion.)

In this report our objective is not simply to obtain the final linearized equations expeditiously; we would also like to explore the challenges of the unrestricted motion problem. For this reason, and to expose the dangers of premature linearization, in the illustration of Kane's method and subsequent methods as well we will refrain from linearizing until we can see clearly what the cost of complete nonlinear analysis would be and at what point (in retrospect) we could have linearized.

Accordingly, we perform inertial time derivatives of Eqs. (269) to obtain $\dot{\mathbf{R}}^0$, $\dot{\mathbf{R}}^1$, and $\dot{\mathbf{R}}^2$. These differentiations are greatly facilitated by the relationships

$$\begin{aligned}\dot{\mathbf{b}}_2^2 &= (\boldsymbol{\omega}^0 + \dot{\gamma}_2\mathbf{b}_1^0) \times \mathbf{b}_2^2 \\ &= (\omega_1 + \dot{\gamma}_2)(\mathbf{b}_3^0 \cos\gamma_2 - \mathbf{b}_2^0 \sin\gamma_2) + \omega_2 \sin\gamma_2 \mathbf{b}_1^0 - \omega_3 \cos\gamma_2 \mathbf{b}_1^0\end{aligned} \tag{270a}$$

$$\begin{aligned}\dot{\mathbf{b}}_2^1 &= (\boldsymbol{\omega}^0 + \dot{\gamma}_1\mathbf{b}_1^0) \times \mathbf{b}_2^1 \\ &= (\omega_1 + \dot{\gamma}_1)(\mathbf{b}_3^0 \cos\gamma_1 - \mathbf{b}_2^0 \sin\gamma_1) + (\omega_2 \sin\gamma_1 - \omega_3 \cos\gamma_1)\mathbf{b}_1^0\end{aligned} \tag{270b}$$

and

$$\dot{\mathbf{b}}_2^0 = \boldsymbol{\omega}^0 \times \mathbf{b}_2^0 = \omega_1\mathbf{b}_3^0 - \omega_3\mathbf{b}_1^0 \tag{270c}$$

The inertial velocities are then after some manipulation given exactly by

$$\begin{aligned}\dot{\mathbf{R}}^0 = R\{&\mathbf{b}_1^0[\omega_2(\sin\gamma_2 - \sin\gamma_1) + \omega_3(\cos\gamma_1 - \cos\gamma_2)] \\ &+ \mathbf{b}_2^0[\omega_1(\sin\gamma_1 - \sin\gamma_2) + \dot{\gamma}_1 \sin\gamma_1 - \dot{\gamma}_2 \sin\gamma_2] \\ &+ \mathbf{b}_3^0[\omega_1(\cos\gamma_2 - \cos\gamma_1) + \dot{\gamma}_2 \cos\gamma_2 - \dot{\gamma}_1 \cos\gamma_1]\}\end{aligned} \tag{271a}$$

$$\begin{aligned}\dot{\mathbf{R}}^1 = &\mathbf{b}_1^0\{\omega_2[(L-R)\sin\gamma_1 + R\sin\gamma_2] - \omega_3[r + (L-R)\cos\gamma_1 + R\cos\gamma_2]\} \\ &+ \mathbf{b}_2^0\{-\omega_1[(L-R)\sin\gamma_1 + R\sin\gamma_2] - (L-R)\dot{\gamma}_1\sin\gamma_1 - R\dot{\gamma}_2\sin\gamma_2\} \\ &+ \mathbf{b}_3^0\{\omega_1[r + (L-R)\cos\gamma_1 + R\cos\gamma_2] + (L-R)\dot{\gamma}_1\cos\gamma_1 + R\dot{\gamma}_2\cos\gamma_2\}\end{aligned} \tag{271b}$$

$$\begin{aligned}\dot{\mathbf{R}}^2 = &\mathbf{b}_1^0\{-\omega_2[(L-R)\sin\gamma_2 + R\sin\gamma_1] + \omega_3[r + (L-R)\cos\gamma_2 + R\cos\gamma_1]\} \\ &+ \mathbf{b}_2^0\{\omega_1[(L-R)\sin\gamma_2 + R\sin\gamma_1] + (L-R)\dot{\gamma}_2\sin\gamma_2 + R\dot{\gamma}_1\sin\gamma_1\} \\ &+ \mathbf{b}_3^0\{-\omega_1[r + (L-R)\cos\gamma_2 + R\cos\gamma_1] - (L-R)\dot{\gamma}_2\cos\gamma_2 - R\dot{\gamma}_1\cos\gamma_1\}\end{aligned} \tag{271c}$$

and in linearized approximation by

$$\dot{\mathbf{R}}^0 \cong R\{\mathbf{b}_1^0\omega_2(\gamma_2 - \gamma_1) + \mathbf{b}_2^0\omega_1(\gamma_1 - \gamma_2) + \mathbf{b}_3^0(\dot{\gamma}_2 - \dot{\gamma}_1)\} \tag{272a}$$

$$\begin{aligned}\dot{\mathbf{R}}^1 \cong &\mathbf{b}_1^0\{\omega_2[(L-R)\gamma_1 + R\gamma_2] - \omega_3(r+L)\} \\ &+ \mathbf{b}_2^0\{-\omega_1[(L-R)\gamma_1 + R\gamma_2]\} + \mathbf{b}_3^0\{\omega_1(r+L) + (L-R)\dot{\gamma}_1 + R\dot{\gamma}_2\}\end{aligned} \tag{272b}$$

$$\begin{aligned}\dot{\mathbf{R}}^2 \cong &\mathbf{b}_1^0\{-\omega_2[(L-R)\gamma_2 + R\gamma_1] + \omega_3(r+L)\} \\ &+ \mathbf{b}_2^0\{\omega_1[(L-R)\gamma_2 + R\gamma_1]\} + \mathbf{b}_3^0\{-\omega_1(r+L) - (L-R)\dot{\gamma}_2 - R\dot{\gamma}_1\}\end{aligned} \tag{272c}$$

We can now compare Eqs. (267) with Eqs. (268) and (271), and (noting the definitions in Eq. (264)) record Kane's coefficient vectors by inspection

$$\mathbf{V}_1^{c_0} = R(\mathbf{b}_2^0 \sin\gamma_1 - \mathbf{b}_3^0 \cos\gamma_1) \cong R(\gamma_1\mathbf{b}_2^0 - \mathbf{b}_3^0) \tag{273a}$$

$$\mathbf{V}_1^{c_1} = -(L-R)\sin\gamma_1\mathbf{b}_2^0 + (L-R)\cos\gamma_1\mathbf{b}_3^0 \cong (L-R)(\mathbf{b}_3^0 - \gamma_1\mathbf{b}_2^0) \qquad (273b)$$

$$\mathbf{V}_1^{c_2} = R(\mathbf{b}_2^0\sin\gamma_1 - \mathbf{b}_3^0\cos\gamma_1) \cong R(\mathbf{b}_2^0\gamma_1 - \mathbf{b}_3^0) \qquad (273c)$$

$$\mathbf{V}_2^{c_0} = R(-\mathbf{b}_2^0\sin\gamma_2 + \mathbf{b}_3^0\cos\gamma_2) \cong R(\mathbf{b}_3^0 - \gamma_2\mathbf{b}_2^0) \qquad (273d)$$

$$\mathbf{V}_2^{c_1} = R(-\mathbf{b}_2^0\sin\gamma_2 + \mathbf{b}_3^0\cos\gamma_2) \cong R(\mathbf{b}_3^0 - \gamma_2\mathbf{b}_2^0) \qquad (273e)$$

$$\mathbf{V}_2^{c_2} = (L-R)(\sin\gamma_2\mathbf{b}_2^0 - \cos\gamma_2\mathbf{b}_3^0) \cong (L-R)(\gamma_2\mathbf{b}_2^0 - \mathbf{b}_3^0) \qquad (273f)$$

$$\mathbf{V}_3^{c_0} = R[\mathbf{b}_2^0(\sin\gamma_1 - \sin\gamma_2) + \mathbf{b}_3^0(\cos\gamma_2 - \cos\gamma_1)] \cong R(\gamma_1 - \gamma_2)\mathbf{b}_2^0 \qquad (273g)$$

$$\begin{aligned}\mathbf{V}_3^{c_1} &= -[(L-R)\sin\gamma_1 + R\sin\gamma_2]\mathbf{b}_2^0 + [r + (L-R)\cos\gamma_1 + R\cos\gamma_2]\mathbf{b}_3^0 \\ &\cong [-L\gamma_1 + R(\gamma_1 - \gamma_2)]\mathbf{b}_2^0 + (r+L)\mathbf{b}_3^0 \end{aligned} \qquad (273h)$$

$$\begin{aligned}\mathbf{V}_3^{c_2} &= [(L-R)\sin\gamma_2 + R\sin\gamma_1]\mathbf{b}_2^0 - [r + (L-R)\cos\gamma_2 + R\cos\gamma_1]\mathbf{b}_3^0 \\ &\cong [L\gamma_2 + R(\gamma_1 - \gamma_2)]\mathbf{b}_2^0 - (r+L)\mathbf{b}_3^0 \end{aligned} \qquad (273i)$$

$$\mathbf{V}_4^{c_0} = R(\sin\gamma_2 - \sin\gamma_1)\mathbf{b}_1^0 \cong R(\gamma_2 - \gamma_1)\mathbf{b}_1^0 \qquad (273j)$$

$$\mathbf{V}_4^{c_1} = [(L-R)\sin\gamma_1 + R\sin\gamma_2]\mathbf{b}_1^0 \cong [L\gamma_1 + R(\gamma_2 - \gamma_1)]\mathbf{b}_1^0 \qquad (273k)$$

$$\mathbf{V}_4^{c_2} = -[(L-R)\sin\gamma_2 + R\sin\gamma_1]\mathbf{b}_1^0 \cong [-L\gamma_2 + R(\gamma_1 - \gamma_2)]\mathbf{b}_1^0 \qquad (273l)$$

$$\mathbf{V}_5^{c_0} = R(\cos\gamma_1 - \cos\gamma_2)\mathbf{b}_1^0 \cong 0 \qquad (273m)$$

$$\mathbf{V}_5^{c_1} = -[r + (L-R)\cos\gamma_1 + R\cos\gamma_2]\mathbf{b}_1^0 \cong -(r+L)\mathbf{b}_1^0 \qquad (273n)$$

$$\mathbf{V}_5^{c_2} = [r + (L-R)\cos\gamma_2 + R\cos\gamma_1]\mathbf{b}_1^0 = (r+L)\mathbf{b}_1^0 \qquad (273o)$$

$$\boldsymbol{\omega}_1^0 = \boldsymbol{\omega}_1^2 = 0 \qquad (273p)$$

$$\boldsymbol{\omega}_1^1 = \mathbf{b}_1^0 \qquad (273q)$$

$$\boldsymbol{\omega}_2^0 = \boldsymbol{\omega}_2^1 = 0 \qquad (273r)$$

$$\boldsymbol{\omega}_2^2 = \mathbf{b}_1^0 \qquad (273s)$$

$$\boldsymbol{\omega}_3^0 = \boldsymbol{\omega}_3^1 = \boldsymbol{\omega}_3^2 = \mathbf{b}_1^0 \qquad (273t)$$

$$\boldsymbol{\omega}_4^0 = \boldsymbol{\omega}_4^1 = \boldsymbol{\omega}_4^2 = \mathbf{b}_2^0 \qquad (273u)$$

$$\boldsymbol{\omega}_5^0 = \boldsymbol{\omega}_5^1 = \boldsymbol{\omega}_5^2 = \mathbf{b}_3^0 \qquad (273v)$$

Note that the linearized approximations recorded in Eqs. (273) could *not* have been obtained correctly from the linearized inertial velocities recorded in Eqs. (272), although they *could* have been obtained from an approximation of Eqs. (271) in which nonlinear terms in $\gamma_1$ and $\gamma_2$ were ignored without imposing any size restrictions on $\dot{\gamma}_1$ and $\dot{\gamma}_2$.

The next step in this procedure is to prepare to utilize Kane's coefficient vectors as required by Eqs. (266); this demands expressions for $\ddot{\mathbf{R}}^j$, $\dot{\mathbf{H}}^j$, $\mathbf{F}^j$, and $\mathbf{M}^j$. Now it is quite clear that these expressions can be linearized immediately in $\gamma_1$, $\gamma_2$, and their time derivatives.

From Eqs. (272), using Eq. (270c) and $\dot{\mathbf{b}}_1^0 = \boldsymbol{\omega}^0 \times \mathbf{b}_1^0$ and $\dot{\mathbf{b}}_3^0 = \boldsymbol{\omega}^0 \times \mathbf{b}_3^0$, differentiation produces the approximations

$$\begin{aligned}\ddot{\mathbf{R}}^0 \cong R\,\{&\mathbf{b}_1^0\,[\dot{\omega}_2(\gamma_2-\gamma_1) + 2\omega_2(\dot{\gamma}_2-\dot{\gamma}_1) + \omega_1\omega_3(\gamma_2-\gamma_1)]\\ &+ \mathbf{b}_2^0\,[\dot{\omega}_1(\gamma_1-\gamma_2) + 2\omega_1(\dot{\gamma}_1-\dot{\gamma}_2) - \omega_2\omega_3(\gamma_1-\gamma_2)]\\ &+ \mathbf{b}_3^0\,[\ddot{\gamma}_2-\ddot{\gamma}_1 + (\omega_1^2+\omega_2^2)(\gamma_1-\gamma_2)]\}\end{aligned} \tag{274a}$$

$$\begin{aligned}\ddot{\mathbf{R}}^1 \cong\; &\mathbf{b}_1^0\,\{\dot{\omega}_2\,[(L-R)\gamma_1 + R\gamma_2] + 2\omega_2\,[(L-R)\dot{\gamma}_1 + R\dot{\gamma}_2] - \dot{\omega}_3(r+L)\\ &+ \omega_1\omega_2(r+L) + \omega_1\omega_3\,[(L-R)\gamma_1 + R\gamma_2]\}\\ &+ \mathbf{b}_2^0\{-\dot{\omega}_1\,[(L-R)\gamma_1 + R\gamma_2] - 2\omega_1\,[(L-R)\dot{\gamma}_1 + R\dot{\gamma}_2]\\ &- (\omega_1^2+\omega_3^2)(r+L) + \omega_2\omega_3\,[(L-R)\gamma_1 + R\gamma_2]\}\\ &+ \mathbf{b}_3^0\,\{\dot{\omega}_1(r+L) + (L-R)\ddot{\gamma}_1 + R\ddot{\gamma}_2\\ &- (\omega_1^2+\omega_2^2)\,[(L-R)\gamma_1 + R\gamma_2] + \omega_2\omega_3(r+L)\}\end{aligned} \tag{274b}$$

$$\begin{aligned}\ddot{\mathbf{R}}^2 \cong\; &\mathbf{b}_1^0\,\{-\dot{\omega}_2\,[(L-R)\gamma_2 + R\gamma_1] + \dot{\omega}_3(r+L) - 2\omega_2\,[(L-R)\dot{\gamma}_2 + R\dot{\gamma}_1]\\ &- \omega_1\omega_2(r+L) - \omega_1\omega_3\,[(L-R)\gamma_2 + R\gamma_1]\}\\ &+ \mathbf{b}_2^0\{\dot{\omega}_1\,[(L-R)\gamma_2 + R\gamma_1] + 2\omega_1\,[(L-R)\dot{\gamma}_2 + R\dot{\gamma}_1]\\ &+ (\omega_1^2+\omega_3^2)(r+L) - \omega_2\omega_3\,[(L-R)\gamma_2 + R\gamma_1]\}\\ &+ \mathbf{b}_3^0\{-\dot{\omega}_1(r+L) - (L-R)\ddot{\gamma}_2 - R\ddot{\gamma}_1\\ &+ (\omega_1^2+\omega_2^2)\,[(L-R)\gamma_2 + R\gamma_1] - \omega_2\omega_3(r+L)\}\end{aligned} \tag{274c}$$

The angular momenta required by Eq. (266b) are

$$\mathbf{H}^0 = \mathbf{I}^0\cdot\boldsymbol{\omega}^0 = I_1^0\omega_1\mathbf{b}_1^0 + I_2^0\omega_2\mathbf{b}_2^0 + I_3^0\omega_3\mathbf{b}_3^0 \tag{275a}$$

(where $I_1^0, I_2^0, I_3^0$ are the mass center principal axis inertias of $\mathscr{s}_0$)

$$\begin{aligned}\mathbf{H}^1 &= \mathbf{I}^1\cdot(\boldsymbol{\omega}^0 + \dot{\gamma}_1\mathbf{b}_1^0) = \frac{mL^2}{3}(\mathbf{b}_1^1\mathbf{b}_1^1 + \mathbf{b}_3^1\mathbf{b}_3^1)\cdot(\boldsymbol{\omega}^0 + \dot{\gamma}_1 b_1^0)\\ &\cong \left(\frac{mL^2}{3}\right)\{[\mathbf{b}_1^0\mathbf{b}_1^0 + \mathbf{b}_3^0\mathbf{b}_3^0 - (\mathbf{b}_2^0\mathbf{b}_3^0 + \mathbf{b}_3^0\mathbf{b}_2^0)\gamma_1]\cdot(\boldsymbol{\omega}^0 + \dot{\gamma}_1 b_1^0)\}\\ &= \left(\frac{mL^2}{3}\right)\{\mathbf{b}_1^0(\omega_1 + \dot{\gamma}_1) - b_2^0\omega_3\gamma_1 + \mathbf{b}_3^0(\omega_3 - \omega_2\gamma_1)\}\end{aligned} \tag{275b}$$

$$\begin{aligned}\mathbf{H}^2 &= \mathbf{I}^2\cdot(\boldsymbol{\omega}^0 + \dot{\gamma}_2\mathbf{b}_1^0) \cong \left(\frac{mL^2}{3}\right)\{[\mathbf{b}_1^0\mathbf{b}_1^0 + \mathbf{b}_3^0\mathbf{b}_3^0 - (\mathbf{b}_2^0\mathbf{b}_3^0 + \mathbf{b}_3^0\mathbf{b}_2^0)\gamma_2]\cdot(\boldsymbol{\omega}^0 + \dot{\gamma}_2\mathbf{b}_1^0)\}\\ &= \left(\frac{mL^2}{3}\right)\{\mathbf{b}_1^0(\omega_1 + \dot{\gamma}_2) - \mathbf{b}_2^0\,\omega_3\gamma_2 + \mathbf{b}_3^0(\omega_3 - \omega_2\gamma_2)\}\end{aligned} \tag{275c}$$

Inertial time differentiation then provides

$$\begin{aligned}\dot{\mathbf{H}}^0 = \mathbf{b}_1^0\,[I^0\dot{\omega}_1 - \omega_2\omega_3(I_2^0 - I_3^0)] + \mathbf{b}_2^0\,[I_2^0\dot{\omega}_2 - \omega_3\omega_1(I_3^0 - I_1^0)]\\ + \mathbf{b}_3^0\,[I_3^0\dot{\omega}_3 - \omega_1\omega_2(I_1^0 - I_2^0)]\}\end{aligned} \tag{276a}$$

$$\dot{\mathbf{H}}^1 = \left(\frac{mL^2}{3}\right)\{\mathbf{b}_1^0\,[\dot{\omega}_1 + \ddot{\gamma}_1 + \omega_2\omega_3 + (\omega_3^2 - \omega_2^2)\,\gamma_1]$$

$$+ \mathbf{b}_2^0\,[-\dot{\omega}_3\gamma_1 + \omega_1\omega_2\gamma_1]$$

$$+ \mathbf{b}_3^0\,[\dot{\omega}_3 - \dot{\omega}_2\gamma_1 - 2\omega_2\dot{\gamma}_1 - \omega_1\omega_3\gamma_1 - \omega_1\omega_2]\} \tag{276b}$$

$$\dot{\mathbf{H}}^2 = \left(\frac{mL^2}{3}\right)\{\mathbf{b}_1^0\,[\dot{\omega}_1 + \ddot{\gamma}_2 + \omega_2\omega_3 + (\omega_3^2 - \omega_2^2)\,\gamma_2]$$

$$+ \mathbf{b}_2^0\,[-\dot{\omega}_3\gamma_2 + \omega_1\omega_2\gamma_2]$$

$$+ \mathbf{b}_3^0\,[\dot{\omega}_3 - \dot{\omega}_2\gamma_2 - 2\omega_2\dot{\gamma}_2 - \omega_1\omega_2 - \dot{\omega}_1\omega_3\gamma_2]\} \tag{276c}$$

We can at last obtain a suitable approximation of $f_k^*(k = 1,\cdots,5)$ from Eq. (266b) by substituting Eqs. (273), (274), and (276). The ensuing calculations finally yield the following results (after substituting the definitions in Eqs. (260))

$$f_1^* = -(\mathcal{M} - 2m)\,\ddot{\mathbf{R}}^0 \cdot \mathbf{V}_1^{c_0} - m\ddot{\mathbf{R}}^1 \cdot V_1^{c_1} - m\ddot{\mathbf{R}}^2 \cdot \mathbf{V}_1^{c_2} - \dot{\mathbf{H}}^0 \cdot \boldsymbol{\omega}_1^0 - \dot{\mathbf{H}}^1 \cdot \boldsymbol{\omega}_1^1 - \dot{\mathbf{H}}^2 \cdot \boldsymbol{\omega}_1^2$$

$$\cong -(\mathcal{M} - 2m)\,R^2\,[\ddot{\gamma}_1 - \ddot{\gamma}_2 - (\omega_1^2 + \omega_2^2)\,(\gamma_1 - \gamma_2)]$$

$$-m\,(L - R)\,\{\dot{\omega}_1\,(r + L) + (L - R)\,\ddot{\gamma}_1 + R\ddot{\gamma}_2 - (\omega_1^2 + \omega_2^2)\,[(L - R)\,\gamma_1 + R\gamma_2]$$

$$+ \omega_2\omega_3\,(r + L) - (r + L)\,(\omega_1^2 + \omega_3^2)\,\gamma_1\}$$

$$+ mR\,\{-\dot{\omega}_1\,(r + L) - (L - R)\,\ddot{\gamma}_2 - R\ddot{\gamma}_1 + (\omega_1^2 + \omega_2^2)\,[(L - R)\,\gamma_2 + R\gamma_1]$$

$$- \omega_2\omega_3\,(r + L) - (r + L)\,(\omega_1^2 + \omega_3^2)\,\gamma_1\}$$

$$- 0 - \left(\frac{mL^2}{3}\right)[\dot{\omega}_1 + \ddot{\gamma}_1 + \omega_2\omega_3 + (\omega_3^2 - \omega_2^2)\,\gamma_1] - 0$$

$$= -(J + \mathcal{M}Rr)\,(\dot{\omega}_1 + \omega_2\omega_3) - (J - \mathcal{M}R^2)\,\ddot{\gamma}_1 - \mathcal{M}R^2\ddot{\gamma}_2$$

$$+ [J\,(\omega_2^2 - \omega_3^2) - \mathcal{M}R^2\,(\omega_1^2 + \omega_2^2) - \mathcal{M}Rr\,(\omega_1^2 + \omega_3^2)]\,\gamma_1$$

$$+ \mathcal{M}R^2\,(\omega_1^2 + \omega_2^2)\,\gamma_2 \tag{277a}$$

$$f_2^* = -(\mathcal{M} - 2m)\,\ddot{\mathbf{R}}^0 \cdot \mathbf{V}_2^{c_0} - m\ddot{\mathbf{R}}^1 \cdot \mathbf{V}_2^{c_1} - m\ddot{\mathbf{R}}^2 \cdot \mathbf{V}_2^{c_2} - \dot{\mathbf{H}}^0 \cdot \boldsymbol{\omega}_2^0 - \dot{\mathbf{H}}^1 \cdot \boldsymbol{\omega}_2^1 - \dot{\mathbf{H}}^2 \cdot \boldsymbol{\omega}_2^2$$

$$\cong (\mathcal{M} - 2m)\,R^2\,[\ddot{\gamma}_1 - \ddot{\gamma}_2 - (\omega_1^2 + \omega_2^2)\,(\gamma - \gamma_2)]$$

$$- mR\,\{\dot{\omega}_1\,(r + L) + (L - R)\,\ddot{\gamma}_1 + R\ddot{\gamma}_2 - (\omega_1^2 + \omega_2^2)\,[(L - R)\,\gamma_1 + R\gamma_2]$$

$$+ \omega_2\omega_3\,(r + L) + (r + L)\,(\omega_1^2 + \omega_3^2)\,\gamma_2\}$$

$$- m\,(L - R)\,\{\dot{\omega}_1\,(r + L) + (L - R)\,\ddot{\gamma}_2 + R\ddot{\gamma}_1 - (\omega_1^2 + \omega_2^2)\,[(L - R)\,\gamma_2 + R\gamma_1]$$

$$+ \omega_2\omega_3\,(r + L) + (r + L)\,(\omega_1^2 + \omega_3^2)\,\gamma_2\}$$

$$- 0 - 0 - \left(\frac{mL^2}{3}\right)[\dot{\omega}_1 + \ddot{\gamma}_2 + \omega_2\omega_3 + (\omega_3^2 - \omega_2^2)\,\gamma_2]$$

$$= -(J + \mathcal{M}Rr)\,\dot{\omega}_1 - \mathcal{M}R^2\ddot{\gamma}_1 - (J + \mathcal{M}R^2)\,\ddot{\gamma}_2 - (J + \mathcal{M}Rr)\,\omega_2\omega_3$$

$$+ [J\,(\omega_2^2 - \omega_3^2) - \mathcal{M}R^2\,(\omega_1^2 + \omega_2^2) - \mathcal{M}Rr\,(\omega_1^2 + \omega_3^2)]\,\gamma_2 + \mathcal{M}R^2\,(\omega_1^2 + \omega_2^2)\,\gamma_1 \tag{277b}$$

$$
\begin{aligned}
f_3^* = & -(\mathcal{M}-2m)\,\ddot{\mathbf{R}}^0\cdot\mathbf{V}_3^{c_0} - m\ddot{\mathbf{R}}^1\cdot\mathbf{V}_3^{c_1} - m\ddot{\mathbf{R}}^2\cdot\mathbf{V}_3^{c_2} - \dot{\mathbf{H}}^0\cdot\boldsymbol{\omega}_3^0 - \dot{\mathbf{H}}^1\cdot\boldsymbol{\omega}_3^1 - \dot{\mathbf{H}}^2\cdot\boldsymbol{\omega}_3^2 \\
\cong & \; 0 - m(r+L)\{\dot{\omega}_1(r+L) + (L-R)\ddot{\gamma}_1 + R\ddot{\gamma}_2 \\
& - (\omega_1^2+\omega_2^2)[(L-R)\gamma_1 + R\gamma_2] + \omega_2\omega_3(r+L)\} \\
& + m(r+L)(\omega_1^2+\omega_3^2)[-L\gamma_1 + R(\gamma_1-\gamma_2)] \\
& - m(r+L)\{\dot{\omega}_1(r+L) + (L-R)\ddot{\gamma}_2 + R\ddot{\gamma}_1 \\
& - (\omega_1^2+\omega_2^2)[(L-R)\gamma_2 + R\gamma_1] + \omega_2\omega_3(r+L)\} \\
& - m(r+L)(\omega_1^2+\omega_3^2)[L\gamma_2 + R(\gamma_1-\gamma_2)] - [I_1^0\dot{\omega}_1 - \omega_2\omega_3(I_2^0 - I_3^0)] \\
& - \left(\frac{mL^2}{3}\right)[\dot{\omega}_1 + \ddot{\gamma}_1 + \omega_2\omega_3 + (\omega_3^2-\omega_2^2)\gamma_1] \\
& - \left(\frac{mL^2}{3}\right)[\dot{\omega}_1 + \ddot{\gamma}_2 + \omega_2\omega_3 + (\omega_3^2-\omega_2^2)\gamma_2] \\
= & -I_1\dot{\omega}_1 - (J+\mathcal{M}Rr)(\ddot{\gamma}_1+\ddot{\gamma}_2) + (I_2-I_3)\,\omega_2\omega_3 + (J+\mathcal{M}Rr)(\omega_2^2-\omega_3^2)(\gamma_1+\gamma_2)
\end{aligned}
\tag{277c}
$$

$$
\begin{aligned}
f_4^* = & -(\mathcal{M}-2m)\,\ddot{\mathbf{R}}^0\cdot\mathbf{V}_4^{c_0} - m\ddot{\mathbf{R}}^1\cdot\mathbf{V}_4^{c_1} - m\ddot{\mathbf{R}}^2\cdot\mathbf{V}_4^{c_2} - \dot{\mathbf{H}}^0\boldsymbol{\omega}_4^0 - \dot{\mathbf{H}}^1\cdot\boldsymbol{\omega}_4^1 - \dot{\mathbf{H}}^2\cdot\boldsymbol{\omega}_4^2 \\
\cong & \; 0 + m(\dot{\omega}_3 - \omega_1\omega_2)(r+L)[L\gamma_1 + R(\gamma_2-\gamma_1)] \\
& + m(\dot{\omega}_3 - \omega_1\omega_2)(r+L)[-L\gamma_2 + R(\gamma_1-\gamma_2)] \\
& - [I_2^0\dot{\omega}_2 - \omega_3\omega_1(I_3^0 - I_1^0)] + \left(\frac{mL^2}{3}\right)(\dot{\omega}_3 - \omega_1\omega_2)(\gamma_1+\gamma_2) \\
= & -I_2\dot{\omega}_2 + (\dot{\omega}_3 - \omega_1\omega_2)(J+\mathcal{M}Rr)(\gamma_1+\gamma_2) + \omega_3\omega_1(I_3-I_1)
\end{aligned}
\tag{277d}
$$

and

$$
\begin{aligned}
f_5^* = & -(\mathcal{M}-2m)\,\ddot{\mathbf{R}}^0\cdot\mathbf{V}_5^{c_0} - m\ddot{\mathbf{R}}^1\cdot\mathbf{V}_5^{c_1} - m\ddot{\mathbf{R}}^2\cdot\mathbf{V}_5^{c_2} - \dot{\mathbf{H}}^0\cdot\boldsymbol{\omega}_5^0 - \dot{\mathbf{H}}^1\cdot\boldsymbol{\omega}_5^1 - \dot{\mathbf{H}}^2\cdot\boldsymbol{\omega}_5^2 \\
\cong & \; 0 + m(r+L)[\dot{\omega}_2 L(\gamma_1+\gamma_2) - 2\dot{\omega}_3(r+L) + 2\omega_2 L(\dot{\gamma}_1+\dot{\gamma}_2) + 2\omega_1\omega_2(r+L) \\
& + \omega_1\omega_3 L(\gamma_1+\gamma_2)] - [I_3^0\dot{\omega}_3 - \omega_1\omega_2(I_1^0 - I_2^0)] \\
& - \left(\frac{mL^2}{3}\right)[2\dot{\omega}_3 - (\dot{\omega}_2 - \omega_1\omega_3)(\gamma_1+\gamma_2) - 2\omega_2(\dot{\gamma}_1+\dot{\gamma}_2) - 2\omega_1\omega_2] \\
= & -I_3\dot{\omega}_3 + (J+\mathcal{M}Rr)(\gamma_1+\gamma_2)\,\dot{\omega}_2 + (I_1-I_2)\,\omega_1\omega_2 \\
& + 2(J+\mathcal{M}Rr)\,\omega_2(\dot{\gamma}_1+\dot{\gamma}_2) + (J+\mathcal{M}Rr)\,\omega_1\omega_3(\gamma_1+\gamma_2)
\end{aligned}
\tag{277e}
$$

In the preceding expressions the mass center principal axis moments of inertia of the undeformed system have been employed, using the relationships

$$
I^1 = I_1^0 + 2m\left(r^2 + 2rL + \frac{4}{3}L^2\right) \tag{278a}
$$

$$
I^2 = I_2^0 \tag{278b}
$$

$$
I^3 = I_3^0 + 2m\left(r^2 + 2rL + \frac{4}{3}L^2\right) \tag{278c}
$$

Having found the generalized inertia forces $f_1^*, \cdots, f_5^*$, we have only to obtain expressions from Eq. (266a) for the generalized active forces $f_1, \cdots, f_5$; corresponding generalized forces are then summed as in Eq. (265) to obtain the final equations of motion.

Although in applying Eq. (266a) we could anticipate the disappearance of nonworking constraint forces from the generalized active forces, and substitute only $\mathbf{M}^0 = k(\gamma_1 + \gamma_2)\mathbf{b}_1^0$ and $\mathbf{M}^1 = -k\gamma_1\mathbf{b}_1^0$ and $\mathbf{M}^2 = -k\gamma_2\mathbf{b}_1^0$ into the formula, it may be more instructive to introduce the constraint forces and watch them disappear from Eq. (266a). Accordingly, we will let $\mathbf{F}^{10}$ and $\mathbf{F}^{20}$ be the forces applied to $\mathcal{B}_0$ by $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, and denote the torques applied to $\mathcal{B}_0$ at the hinge connections to $\mathcal{B}_1$ and $\mathcal{B}_2$ by $k\gamma_1\mathbf{b}_1^0 + \tau_2^1\mathbf{b}_2^0 + \tau_3^1\mathbf{b}_3^0$ and $k\gamma_2\mathbf{b}_1^0 + \tau_2^2\mathbf{b}_2^0 + \tau_3^2\mathbf{b}_3^0$ respectively. Then Eq. (266a) provides, for example,

$$\begin{aligned}
f_1 &= \mathbf{F}^0\cdot\mathbf{V}_1^{c_0} + \mathbf{F}^1\cdot\mathbf{V}_1^{c_1} + \mathbf{F}^2\cdot\mathbf{V}_1^{c_2} + \mathbf{M}^0\cdot\boldsymbol{\omega}_1^0 + \mathbf{M}^1\cdot\boldsymbol{\omega}_1^1 + \mathbf{M}^2\cdot\boldsymbol{\omega}_1^2 \\
&= \mathbf{F}^{10}\cdot(\mathbf{V}_1^{c_0} - \mathbf{V}_1^{c_1}) + \mathbf{F}^{20}\cdot(\mathbf{V}_1^{c_0} - \mathbf{V}_1^{c_2}) \\
&\quad + (k\gamma_1\mathbf{b}_1^0 + \tau_2^1\mathbf{b}_2^0 + \tau_3^1\mathbf{b}_3^0)\cdot(\boldsymbol{\omega}_1^0 - \boldsymbol{\omega}_1^1) \\
&\quad + (k\gamma_2\mathbf{b}_1^0 + \tau_2^2\mathbf{b}_2^0 + \tau_3^2\mathbf{b}_3^0)\cdot(\boldsymbol{\omega}_1^0 - \boldsymbol{\omega}_1^2) \\
&\quad + r\mathbf{b}_2^0\times\mathbf{F}^{10}\cdot\boldsymbol{\omega}_1^0 + L\mathbf{b}_2^1\times\mathbf{F}^{10}\cdot\boldsymbol{\omega}_1^1 \\
&\quad - r\mathbf{b}_2^0\times\mathbf{F}^{20}\cdot\boldsymbol{\omega}_1^0 - L\mathbf{b}_2^2\times\mathbf{F}^{20}\cdot\boldsymbol{\omega}_1^2 \\
&= \mathbf{F}^{10}\cdot L\mathbf{b}_3^1 - k\gamma_1 + L\mathbf{b}_2^1\times\mathbf{F}^{10}\cdot\mathbf{b}_1^1 \\
&= -k\gamma_1 + L(\mathbf{F}^{10}\cdot\mathbf{b}_3^1 + \mathbf{b}_1^1\times\mathbf{b}_2^1\cdot\mathbf{F}^{10}) = -k\gamma_1
\end{aligned} \tag{279a}$$

and similarly

$$f_2 = -k\gamma_2 \tag{279b}$$

and

$$f_3 = f_4 = f_5 = 0 \tag{279c}$$

When Eqs. (279) and (277) are combined as required by Eq. (265), the resulting equations of motion are *identical* to those previously obtained from a Newton-Euler formulation and recorded as Eqs. (259). Thus neither procedure has a computational advantage over the other; any advantages must be claimed on the basis of ease of formulation. Before weighing this issue, we proceed to derive equations of motion from yet another point of view.

*c. Lagrange's quasi-coordinate equations.* Our derivation of the general theory of Lagrange's quasi-coordinate equations culminated in Eq. (147), which appears rather formidable when it is to be applied by hand calculation to the three body system in Fig. 7, with as many as eight degrees of freedom (six for the central rigid body $\mathcal{B}_0$ and one for each of the rotating appendages). Both to reduce the labors of this example and to illustrate results of general theoretical value, we shall establish some operational short-cuts for the application of Eq. (147).

It should first be recalled that in the absence of external forces the system mass center is inertially stationary. If instead of working with translation coordinates for $\mathcal{B}_0$ we adopt the (constant) coordinates of the system mass center, we can reduce the system to one having five degrees of freedom.

The obvious choice for a full set of quasi-coordinates is that displayed in Eq. (264). A more subtle choice would be to replace $u_1$ and $u_2$ by $\dot{\eta}_1$ and $\dot{\eta}_2$ as defined by Eqs. (262); this option, which was also available with Kane's approach, we forego for the present.

It appears from the derivation preceding Eq. (147) that we must next introduce some set of generalized coordinates $q_1$, $q_2$, $q_3$ describing the inertial attitude of $\mathscr{b}_0$, to construct the 5 by 5 matrix $W$ appearing in Eq. (147) and defined by Eq. (134). There would follow the inversion of $W$, the determination of $W_{ij,q}$ and $W_{,q_k}$ ($W_{,t}$ being zero), and the whole painful process already illustrated in the two preceding examples. This labor is particularly unpleasant in this example because we know from the Newton-Euler formulation or Kane's equations that inertial attitude angles of $\mathscr{b}_0$ have no place in the equations of motion. It can't make any difference which set of attitude angles we choose; they must eventually all disappear when we grind out the details of Eq. (147).

**Fortunately, we can markedly reduce the labors of dealing with $W$ whenever we choose three of the quasi-coordinate derivatives to match the scalar components of the inertial angular velocity of some rigid body or reference frame for a vector basis fixed in that body or frame, and choose the remaining quasi-coordinates as simply the remaining generalized coordinates. With this choice, we always have a matrix $W$ with the structure**

$$W = \left[\begin{array}{c|c} U & 0 \\ \hline 0 & W_0 \end{array}\right] \tag{280}$$

**where $W_0$ is a 3 by 3 matrix. Thus the inverse of $W$ is simply**

$$W = \left[\begin{array}{c|c} U & 0 \\ \hline 0 & W_0^{-1} \end{array}\right] \tag{281}$$

With this choice of quasi-coordinates, the matrix $W^{-1}\gamma$ in Eq. (147b) always becomes

$$W^{-1}\gamma = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \tilde{\omega} \end{array}\right] \tag{282}$$

as indicated for the rigid body in Eq. (198). (To prove this contention, merely repeat the development leading to Eq. (198) in the example, and note that the purely kinematical and mathematical steps in that derivation never utilize the single rigid body restriction of the example.)

With these observations, we discover that Eq. (147b) is most clearly presented as the two distinct matrix equations

$$\frac{d}{dt}(\overline{T}_{,\dot{q}^1}) - \overline{T}_{,q^1} = Q^1 - A_1^T\lambda \tag{283}$$

and

$$\frac{d}{dt}(\overline{T}_{,\omega}) + \tilde{\omega}\overline{T}_{,\omega} - W_0^{-1}\overline{T}_{,q^0} = W_0^{-1}Q^0 - W_0^{-1}A_2^T\lambda \tag{284}$$

where the matrix $A$ defining the $m$ constraints has been partitioned as

$$A = [A_1 \mid A_2]$$

with $A_2$ of dimension $m$ by 3, and the generalized force and coordinate matrices have similarly been partitioned as

$$Q = \{Q^{1^T} \mid Q^{0^T}\}^T \text{ and } q = \{q^{1^T} \mid q^{0^T}\}^T$$

The effect of this separation is to permit the use of Lagrange's quasi-coordinate equations (Eq. (284)) where they are most powerful, and to remain with the generalized coordinate equations (Eq. (283)) where the quasi-coordinate formulation becomes a burden.

In application to the three-body system in Fig. 7, we obviously have $\lambda = 0$, and since we know that the attitude variables for $\mathscr{b}_0$ cannot enter the problem we anticipate that $\overline{T}_{,q_0} = 0$ as well. The matrix $q^1$ has the elements $\gamma_1$ and $\gamma_2$ (or if we prefer, $\eta_1$ and $\eta_2$).

The system kinetic energy is by definition

$$T = \frac{1}{2}\int \dot{\boldsymbol{\rho}} \cdot \dot{\boldsymbol{\rho}}\, dm \tag{285}$$

where $\boldsymbol{\rho}$ is the position vector of a generic mass element from the system mass center CM, which is inertially stationary. It is convenient to replace $\boldsymbol{\rho}$ by the sum $\mathbf{p} + \mathbf{c}$, where $\mathbf{c}$ is the vector from CM to the point O fixed in $\mathscr{b}_0$ and occupied by CM when $\gamma_1 = \gamma_2 = 0$, and $\mathbf{p}$ is the generic position vector from O.

Making use of the mass center defining requirement

$$\int \boldsymbol{\rho}\, dm = \int (\mathbf{p} + \mathbf{c})\, dm = 0 \tag{286}$$

we can write

$$\begin{aligned}
2T &= \int (\dot{\mathbf{p}} + \dot{\mathbf{c}}) \cdot (\dot{\mathbf{p}} + \dot{\mathbf{c}})\, dm = \int \dot{\mathbf{p}} \cdot (\dot{\mathbf{p}} + \dot{\mathbf{c}})\, dm \\
&= \int \dot{\mathbf{p}} \cdot \dot{\mathbf{p}}\, dm + \dot{\mathbf{c}} \cdot \int \dot{\mathbf{p}}\, dm \\
&= \int (\mathring{\mathbf{p}} + \boldsymbol{\omega} \times \mathbf{p}) \cdot (\mathring{\mathbf{p}} + \boldsymbol{\omega} \times \mathbf{p})\, dm - \mathscr{M}\dot{\mathbf{c}} \cdot \dot{\mathbf{c}} \\
&= \int \mathring{\mathbf{p}} \cdot \mathring{\mathbf{p}}\, dm + 2\int \mathring{\mathbf{p}} \cdot (\boldsymbol{\omega} \times \mathbf{p})\, dm + \boldsymbol{\omega} \times \left[ \int \mathbf{p} \cdot (\boldsymbol{\omega} \times \mathbf{p})\, dm \right] \\
&= \mathscr{M} (\mathring{\mathbf{c}} + \boldsymbol{\omega} \times \mathbf{c}) \cdot (\mathring{\mathbf{c}} + \boldsymbol{\omega} \times \mathbf{c})
\end{aligned} \tag{287}$$

where a circle over a vector implies time differentiation in the reference frame established by $\mathscr{b}_0$, and $\boldsymbol{\omega}$ is the inertial angular velocity of $\mathscr{b}_0$. Among the terms in

Eq. (287) is one that simplifies as follows:

$$\begin{aligned}
\boldsymbol{\omega} \times \left[ \int \mathbf{p} \cdot (\boldsymbol{\omega} \times \mathbf{p})\, dm \right] &= \boldsymbol{\omega} \cdot \int \mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p})\, dm \\
&= \boldsymbol{\omega} \cdot \int [\mathbf{p} \cdot \mathbf{p}\boldsymbol{\omega} - \mathbf{p}\mathbf{p} \cdot \boldsymbol{\omega}]\, dm \\
&= \boldsymbol{\omega} \cdot \int [\mathbf{p} \cdot \mathbf{p}\mathbf{U} - \mathbf{p}\mathbf{p}]\, dm \cdot \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega}
\end{aligned} \tag{288}$$

where $\mathbf{U}$ is the unit dyadic and $\mathbf{I}$ is the (time-varying) inertia dyadic of the system for point O.

For all differential mass elements within $\mathscr{b}_0$, the term $\mathring{\mathbf{p}}$ is zero, while for those within $\mathscr{b}_1$ and $\mathscr{b}_2$ $\mathbf{p}$ is simply a vector cross product. Thus the first two system integrals in Eq. (287) can be replaced by single body integrals as follows

$$\begin{aligned}
\int \mathring{\mathbf{p}} \cdot \mathring{\mathbf{p}}\, dm + 2 \int \mathring{\mathbf{p}} \cdot (\boldsymbol{\omega} \times \mathbf{p})\, dm &= \int_0^{2L} (\dot{\gamma}_1 \mathbf{b}_1^1 \times y\mathbf{b}_2^1) \cdot (\dot{\gamma}_1 \mathbf{b}_1^1 \times y\mathbf{b}_2^1)\, \mu\, dy \\
&\quad + \int_0^{2L} [\dot{\gamma}_2 \mathbf{b}_1^2 \times (-y\mathbf{b}_2^2)] \cdot [\dot{\gamma}_2 \mathbf{b}_1^2 \times (-y\mathbf{b}_2^2)]\, \mu\, dy \\
&\quad + 2 \int_0^{2L} (\dot{\gamma}_1 \mathbf{b}_1^1 \times y\mathbf{b}_2^1) \cdot [\boldsymbol{\omega} \times (r\mathbf{b}_2^0 + y\mathbf{b}_2^1)]\, \mu\, dy \\
&\quad + 2 \int_0^{2L} [\dot{\gamma}_2 \mathbf{b}_1^2 \times (-y\mathbf{b}_2^2)] \cdot [\boldsymbol{\omega} \times \\
&\qquad\qquad (-r\mathbf{b}_2^0 - y\mathbf{b}_2^2)]\, \mu\, dy \\
&= \int_0^{2L} (\dot{\gamma}_1^2 + \dot{\gamma}_2^2)\, y^2 \mu\, dy \\
&\quad + 2 \int_0^{2L} \{\dot{\gamma}_1 y\mathbf{b}_3^1 \cdot [\boldsymbol{\omega} \times (r\mathbf{b}_2^0 + y\mathbf{b}_2^1)] \\
&\quad + \dot{\gamma}_2 y\mathbf{b}_3^2 \cdot [\boldsymbol{\omega} \times (r\mathbf{b}_2^0 + y\mathbf{b}_2^2)]\}\, \mu\, dy \\
&= J(\dot{\gamma}_1^2 + \dot{\gamma}_2^2) \\
&\quad + 2\,[\dot{\gamma}_1 (\mathbf{b}_3^0 \cos\gamma_1 - \mathbf{b}_2^0 \sin\gamma_1) \\
&\quad + \gamma_2 (\mathbf{b}_3^0 \cos\gamma_2 - \mathbf{b}_2^0 \sin\gamma_2)] \cdot (\omega_1 r\mathbf{b}_3^0 - \omega_3 r\mathbf{b}_1^0)\, \mu \\
&\quad \times \int_0^{2L} y\, dy + 2\dot{\gamma}_1 (\mathbf{b}_3^0 \cos\gamma_1 - \mathbf{b}_2^0 \sin\gamma_1) \\
&\quad \cdot [\boldsymbol{\omega} \times (\mathbf{b}_2^0 \cos\gamma_1 + \mathbf{b}_3^0 \sin\gamma_1)]\, \mu \\
&\quad \times \int_0^{2L} y^2\, dy + 2\dot{\gamma}_2 (\mathbf{b}_3^0 \cos\gamma_2 - \mathbf{b}_2^0 \sin\gamma_2) \\
&\quad \cdot [\boldsymbol{\omega} \times (\mathbf{b}_2^0 \cos\gamma_2 + \mathbf{b}_3^0 \sin\gamma_2)]\, \mu \int_0^{2L} y^2\, dy
\end{aligned}$$

$$
\begin{aligned}
&= J(\dot{\gamma}_1^2 + \dot{\gamma}_2^2) + 2mrL\omega_1(\dot{\gamma}_1 \cos\gamma_1 + \dot{\gamma}_2 \cos\gamma_2) \\
&\quad + 2J[\dot{\gamma}_1\omega_1(\sin^2\gamma_1 + \cos^2\gamma_1) \\
&\quad + \dot{\gamma}_2\omega_1(\sin^2\gamma_2 + \cos^2\gamma_2)] \\
&= J(\dot{\gamma}_1^2 + \dot{\gamma}_2^2) + 2\dot{\gamma}_1\omega_1(J + mrL\cos\gamma_1) \\
&\quad + 2\dot{\gamma}_2\omega_1(J + mrL\cos\gamma_2) \qquad (289)
\end{aligned}
$$

(Recall the definition of $J$ in Eq. (260b), and the definition $m \stackrel{\Delta}{=} 2\mu L$.)

Now we can return to Eq. (287) and expand the final term. The basic ingredient of this term is $\mathbf{c}$, which by mass center definition satisfies the equation

$$
\begin{aligned}
-\mathcal{M}\mathbf{c} &= m(r\mathbf{b}_2^0 + L\mathbf{b}_2^1) + m(-r\mathbf{b}_2^0 - L\mathbf{b}_2^2) \\
&= mL[\mathbf{b}_2^0(\cos\gamma_1 - \cos\gamma_2) + \mathbf{b}_3^0(\sin\gamma_1 - \sin\gamma_2)] \qquad (290)
\end{aligned}
$$

With the implications

$$
\overset{\circ}{\mathbf{c}} = [\dot{\gamma}_1(\sin\gamma_1\mathbf{b}_2^0 - \cos\gamma_1\mathbf{b}_3^0) - \dot{\gamma}_2(\sin\gamma_2\mathbf{b}_2^0 - \cos\gamma_2 b_3^0)]\, mL/\mathcal{M} \qquad (291)
$$

and

$$
\begin{aligned}
\boldsymbol{\omega} \times \mathbf{c} = \{&\mathbf{b}_1^0[\omega_3(\cos\gamma_1 - \cos\gamma_2) - \omega_2(\sin\gamma_1 - \sin\gamma_2)] \\
&+ \mathbf{b}_2^0[\omega_1(\sin\gamma_1 - \sin\gamma_2)] - \mathbf{b}_3^0[\omega_1(\cos\gamma_1 - \cos\gamma_2)]\}\, mL/\mathcal{M}
\end{aligned}
$$
$$(292)$$

we can expand the final term in Eq. (287) as

$$
\begin{aligned}
-\mathcal{M}(\overset{\circ}{\mathbf{c}} + \boldsymbol{\omega}\times\mathbf{c})\cdot(\overset{\circ}{\mathbf{c}} + \boldsymbol{\omega}\times\mathbf{c}) = -\{&(\dot{\gamma}_1\sin\gamma_1 - \dot{\gamma}_2\sin\gamma_2)^2 + (\dot{\gamma}_2\cos\gamma_2 - \dot{\gamma}_1\cos\gamma_1)^2 \\
&+ [\omega_3(\cos\gamma_1 - \cos\gamma_2) - \omega_2(\sin\gamma_1 - \sin\gamma_2)]^2 \\
&+ \omega_1^2[(\sin\gamma_1 - \sin\gamma_2)^2 + (\cos\gamma_1 - \cos\gamma_2)^2] \\
&+ 2\omega_1(\sin\gamma_1 - \sin\gamma_2)(\dot{\gamma}_1\sin\gamma_1 - \dot{\gamma}_2\sin\gamma_2) \\
&+ 2\omega_1(\cos\gamma_1 - \cos\gamma_2)(\dot{\gamma}_1\cos\gamma_1 - \dot{\gamma}_2\cos\gamma_2)\} \\
&\times m^2L^2/\mathcal{M} \qquad (293)
\end{aligned}
$$

Finally we can combine Eqs. (287), (288), (289), and (293) into an expression for $T$ that is exact and very nearly explicit in its dependence on the kinematic variables. In transition we replace the vector-dyadic product $\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega}$ by its matrix counterpart $\omega^T I\omega$, adopting the vector basis established by $\mathbf{b}_1^0$, $\mathbf{b}_2^0$, and $\mathbf{b}_3^0$. The result is (after slight simplification of Eq. (293))

$$
\begin{aligned}
2T = \omega^T I\omega &+ J(\dot{\gamma}_1^2 + \dot{\gamma}_2^2) + 2\omega_1[\dot{\gamma}_1(J + mrL\cos\gamma_1) + \dot{\gamma}_2(J + mrL\cos\gamma_2)] \\
&- \{\dot{\gamma}_1^2 + \dot{\gamma}_2^2 - 2\dot{\gamma}_1\dot{\gamma}_2(\cos\gamma_1\cos\gamma_2 + \sin\gamma_1\sin\gamma_2) \\
&+ 2\omega_1[(\dot{\gamma}_1\sin\gamma_1 - \dot{\gamma}_2\sin\gamma_2)(\sin\gamma_1 - \sin\gamma_2) \\
&+ (\dot{\gamma}_1\cos\gamma_1 - \dot{\gamma}_2\cos\gamma_2)(\cos\gamma_1 - \cos\gamma_2)] \\
&+ 2\omega_1^2(1 - \cos\gamma_1\cos\gamma_2 - \sin\gamma_1\sin\gamma_2) \\
&+ [\omega_3(\cos\gamma_1 - \cos\gamma_2) - \omega_2(\sin\gamma_1 - \sin\gamma_2)]^2\}\, m^2L^2/\mathcal{M} \qquad (294)
\end{aligned}
$$

To make $T$ fully explicit in terms of $\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \dot{\gamma}_1$, and $\dot{\gamma}_2$ as required by Lagrange's equations, we must recognize that $\omega \triangleq \{\omega_1 \omega_2 \omega_3\}^T$ and that the inertia matrix $I$ in vector basis $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$ is given by

$$I = \int (p^T p U - p p^T)\, dm \tag{295}$$

where $p$ represents $\mathbf{p}$ in basis $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$. We can allow $I_1^0, I_2^0, I_3^0$ to represent the (principal) moments of inertia of $\mathscr{B}_0$ in this basis, and let $I^0$ be the corresponding inertia matrix of $\mathscr{B}_0$; then we have

$$I = I^0 + \int_{\mathscr{B}_1} (p^T p U - p p^T)\, dm + \int_{\mathscr{B}_2} (p^T p U - p p^T)\, dm \tag{296}$$

The first of these integrals is available from the dyadic

$$\begin{aligned}
\int_{\mathscr{B}_1} (\mathbf{p} \cdot \mathbf{p}\mathbf{U} - \mathbf{p}\mathbf{p})\, dm &= \int_0^{2L} [(r\mathbf{b}_2^0 + y\mathbf{b}_2^1) \cdot (r\mathbf{b}_2^0 + y\mathbf{b}_2^1)\, \mathbf{U} \\
&\quad - (r\mathbf{b}_2^0 + y\mathbf{b}_2^1)(r\mathbf{b}_2^0 + y\mathbf{b}^1)]\, \mu\, dy \\
&= \int_0^{2L} \{[r\mathbf{b}_2^0 + y(\mathbf{b}_2^0 \cos\gamma_1 + \mathbf{b}_3^0 \sin\gamma_1)] \\
&\quad \cdot [r\mathbf{b}_2^0 + y(\mathbf{b}_2^0 \cos\gamma_1 + \mathbf{b}_3^0 \sin\gamma_1)]\, \mathbf{U} \\
&\quad - [r\mathbf{b}_2^0 + y(\mathbf{b}_2^0 \cos\gamma_1 + \mathbf{b}_3^0 \sin\gamma_1)][r\mathbf{b}_2^0 \\
&\quad + y(\mathbf{b}_2^0 \cos\gamma_1 + \mathbf{b}_3^0 \sin\gamma_1)]\}\, \mu\, dy \\
&= mr^2 (\mathbf{U} - \mathbf{b}_2^0\mathbf{b}_2^0) + J(\mathbf{U} - \mathbf{b}_2^0\mathbf{b}_2^0 \cos^2\gamma_1 - \mathbf{b}_3^0\mathbf{b}_3^0 \sin^2\gamma_1 \\
&\quad - \mathbf{b}_2^0\mathbf{b}_3^0 \sin\gamma_1 \cos\gamma_1 - \mathbf{b}_3^0\mathbf{b}_2^0 \sin\gamma_1 \cos\gamma_1) \\
&\quad + mrL(2\cos\gamma_1 \mathbf{U} - 2\mathbf{b}_2^0\mathbf{b}_2^0 \cos\gamma_1 - \mathbf{b}_2^0\mathbf{b}_3^0 \sin\gamma_1 \\
&\quad - \mathbf{b}_3^0\mathbf{b}_2^0 \sin\gamma_1) \\
&= (mr^2 + J + 2mrL)(\mathbf{U} - \mathbf{b}_2^0\mathbf{b}_2^0) \\
&\quad + J(\mathbf{b}_2^0\mathbf{b}_2^0 \sin^2\gamma_1 - \mathbf{b}_3^0\mathbf{b}_3^0 \sin^2\gamma_1 - \mathbf{b}_2^0\mathbf{b}_3^0 \sin\gamma_1 \cos\gamma_1 \\
&\quad - \mathbf{b}_3^0\mathbf{b}_2^0 \sin\gamma_1 \cos\gamma_1) - mrL[2(1 - \cos\gamma_1)\, \mathbf{U} \\
&\quad + 2\mathbf{b}_2^0\mathbf{b}_2^0 \cos\gamma_1 + \mathbf{b}_2^0\mathbf{b}_3^0 \sin\gamma_1 + \mathbf{b}_3^0\mathbf{b}_2^0 \sin\gamma_1]
\end{aligned} \tag{297}$$

The second integral in Eq. (296) corresponds to a dyadic represented by Eq. (297) with $\gamma_2$ replacing $\gamma_1$.

Now we can return to Eq. (296) and record the elements of the symmetric inertia matrix $I$ explicitly, as follows:

$$I_{11} = I_1^0 + 2(mr^2 + J + 2mrL) - 2mrL(2 - \cos\gamma_1 - \cos\gamma_2) \tag{298a}$$

$$I_{22} = I_2^0 + J(\sin^2\gamma_1 + \sin^2\gamma_2) \tag{298b}$$

$$\begin{aligned}
I_{33} &= I_3^0 + 2(mr^2 + J + 2mrL) - J(\sin^2\gamma_1 + \sin^2\gamma_2) \\
&\quad - 2mrL(2 - \cos\gamma_1 - \cos\gamma_2)
\end{aligned} \tag{298c}$$

$$I_{12} = 0 \tag{298d}$$

$$I_{13} = 0 \tag{298e}$$

$$I_{23} = -J(\sin\gamma_1\cos\gamma_1 + \sin\gamma_2\cos\gamma_2) - mrL(\sin\gamma_1 + \sin\gamma_2) \tag{298f}$$

The values of the inertia matrix elements that survive when $\gamma_1 = \gamma_2 = 0$ are the principal axis inertias of the system in its nominal configuration about the system mass center; these we can call $I_1$, $I_2$, and $I_3$, and we can rewrite Eqs. (298a through c) as

$$I_{11} = I_1 - 2mrL(2 - \cos\gamma_1 - \cos\gamma_2) \tag{298g}$$

$$I_{22} = I_2 + J(\sin^2\gamma_1 + \sin^2\gamma_2) \tag{298h}$$

$$I_{33} = I_3 - J(\sin^2\gamma_1 + \sin^2\gamma_2) - 2mrL(2 - \cos\gamma_1 - \cos\gamma_2) \tag{298i}$$

Finally we can return to Eq. (294) and record $T$ in explicit scalar form as

$$\begin{aligned}
T = {} & \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + \frac{1}{2}(J - \mathcal{M}R^2)(\dot\gamma_1^2 + \dot\gamma_2^2) \\
& + \mathcal{M}R^2\dot\gamma_1\dot\gamma_2(\cos\gamma_1\cos\gamma_2 + \sin\gamma_1\sin\gamma_2) \\
& + \omega_1\{(\dot\gamma_1 + \dot\gamma_2)J + \dot\gamma_1\mathcal{M}Rr\cos\gamma_1 + \dot\gamma_2\mathcal{M}Rr\cos\gamma_2 \\
& - \mathcal{M}R^2[(\dot\gamma_1\sin\gamma_1 - \dot\gamma_2\sin\gamma_2)(\sin\gamma_1 - \sin\gamma_2) \\
& + (\dot\gamma_1\cos\gamma_1 - \dot\gamma_2\cos\gamma_2)(\cos\gamma_1 - \cos\gamma_2)]\} \\
& - \mathcal{M}R^2\Big\{\omega_1^2(1 - \cos\gamma_1\cos\gamma_2 - \sin\gamma_1\sin\gamma_2) + \frac{1}{2}[\omega_3(\cos\gamma_1 - \cos\gamma_2) \\
& - \omega_2(\sin\gamma_1 - \sin\gamma_2)]^2\Big\} - \mathcal{M}rR(2 - \cos\gamma_1 - \cos\gamma_2)(\omega_1^2 + \omega_3^2) \\
& + \frac{1}{2}J(\sin^2\gamma_1 + \sin^2\gamma_2)(\omega_2^2 - \omega_3^2) \\
& - \omega_2\omega_3[J(\sin\gamma_1\cos\gamma_1 + \sin\gamma_2\cos\gamma_2) + \mathcal{M}Rr(\sin\gamma_1 + \sin\gamma_2)]
\end{aligned} \tag{299}$$

in which we have replaced $mL$ by $\mathcal{M}R$, in accordance with the definition appearing in Eq. (260a).

The task demanded by Lagrange's equations in the form of Eqs. (283) and (284) is the partial differentiation of $\bar{T}$ with respect to $\omega_1$, $\omega_2$, $\omega_3$, $\gamma_1$, $\gamma_2$, $\dot\gamma_1$, and $\dot\gamma_2$. As we have expressed $\bar{T}$ in Eq. (299), this appears to be a formidable undertaking. (Remember that our example is a very simple symmetric system of three rigid bodies; the derivation of Lagrange's equations of motion for the unrestricted motion of a general set of even six or eight bodies would become a most dreary endeavor!)

But for this example we have agreed to examine only the restricted case in which only first degree terms in $\gamma_1$, $\gamma_2$, $\dot\gamma_1$, and $\dot\gamma_2$ are retained in the equations of motion. For the Newton-Euler formulation, this restriction meant that we could omit terms above the first degree in these variables *wherever* they occurred, replacing $\sin\gamma_1$ by $\gamma_1$ and $\cos\gamma_1$ by 1 from the very outset; a glance at the elements of the inertia matrix I in Eq. (298) suggests the degree of simplification that follows from a procedure of "linearization as you go" in deriving equations of motion.

In a Lagrangian formulation, however, one must not linearize prematurely; second degree terms in $\gamma_1$ appearing in $I$ in Eq. (298) become first degree terms in the equations of motion after the partial differentiation by $\gamma_1$ indicated by Eq. (283). This means that the most that we can do in approximating $\overline{T}$ in Eq. (299) is to reduce it to a quadratic approximation in $\gamma_1$, $\gamma_2$, $\dot{\gamma}_1$, and $\dot{\gamma}_2$. With the substitutions

$$\sin\gamma_1 \cong \gamma_1 \qquad \cos\gamma_1 \cong 1 - \gamma_1^2/2$$

$$\sin\gamma_2 \cong \gamma_2 \qquad \cos\gamma_2 \cong 1 - \gamma_2^2/2$$

and the truncation of terms above the second degree, $\overline{T}$ in Eq. (299) becomes

$$\begin{aligned} T \cong {} & \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + \frac{1}{2}(J - \mathcal{M}R^2)(\dot{\gamma}_1^2 + \dot{\gamma}_2^2) + \mathcal{M}R^2\dot{\gamma}_1\dot{\gamma}_2 \\ & + (J + \mathcal{M}Rr)(\dot{\gamma}_1 + \dot{\gamma}_2)\,\omega_1 - \frac{1}{2}\mathcal{M}R^2(\gamma_1 - \gamma_2)^2(\omega_1^2 + \omega_2^2) \\ & - \frac{1}{2}\mathcal{M}Rr(\gamma_1^2 + \gamma_2^2)(\omega_1^2 + \omega_3^2) - \mathcal{M}Rr(\gamma_1 + \gamma_2)\,\omega_2\omega_3 \\ & + \frac{1}{2}J(\gamma_1^2 + \gamma_2^2)(\omega_2^2 - \omega_3^2) - J(\gamma_1 + \gamma_2)\,\omega_2\omega_3 \end{aligned} \tag{300}$$

Finally the labors of partial and ordinary differentiation can begin. Modest effort produces the differentiated kinetic energy matrices

$$\overline{T}_{,\dot{q}^1} = \begin{Bmatrix} \overline{T}_{,\dot{\gamma}_1} \\ \overline{T}_{,\dot{\gamma}_2} \end{Bmatrix} \cong \begin{Bmatrix} (J - \mathcal{M}R^2)\dot{\gamma}_1 + \mathcal{M}R^2\dot{\gamma}_2 + (J + \mathcal{M}Rr)\,\omega_1 \\ (J - \mathcal{M}R^2)\dot{\gamma}_2 + \mathcal{M}R^2\dot{\gamma}_1 + (J + \mathcal{M}Rr)\,\omega_1 \end{Bmatrix}$$

$$\overline{T}_{,\omega} = \begin{Bmatrix} \overline{T}_{,\omega_1} \\ \overline{T}_{,\omega_2} \\ \overline{T}_{,\omega_3} \end{Bmatrix} \cong \begin{Bmatrix} I_1\omega_1 + (J + \mathcal{M}Rr)(\dot{\gamma}_1 + \dot{\gamma}_2) \\ I_2\omega_2 - (J + \mathcal{M}Rr)(\gamma_1 + \gamma_2)\,\omega_3 \\ I_3\omega_3 - (J + \mathcal{M}Rr)(\gamma_1 + \gamma_2)\,\omega_2 \end{Bmatrix}$$

$$\overline{T}_{,q^1} = \begin{Bmatrix} \overline{T}_{,\gamma_1} \\ \overline{T}_{,\gamma_2} \end{Bmatrix} \cong \begin{Bmatrix} -\mathcal{M}R^2(\gamma_1 - \gamma_2)(\omega_1^2 + \omega_2^2) - \mathcal{M}Rr\gamma_1(\omega_1^2 + \omega_3^2) - \mathcal{M}Rr\omega_2\omega_3 \\ + J\gamma_1(\omega_2^2 - \omega_3^2) - J\omega_2\omega_3 \\ \mathcal{M}R^2(\gamma_1 - \gamma_2)(\omega_1^2 + \omega_2^2) - \mathcal{M}Rr\gamma_2(\omega_1^2 + \omega_3^2) - \mathcal{M}Rr\omega_2\omega_3 \\ + J\gamma_2(\omega_2^2 - \omega_3^2) - J\omega_2\omega_3 \end{Bmatrix}$$

$$\overline{T}_{,q^0} = \{0\}$$

The generalized force matrices in Eqs. (283) and (284) become

$$Q^1 = \begin{Bmatrix} -k\gamma_1 \\ -k\gamma_2 \end{Bmatrix}, \qquad Q^0 = \{0\}$$

Since $\lambda = 0$ in Eqs. (283) and (284), all that remains is the time differentiation of $\overline{T}_{,\dot{q}^1}$ and $\overline{T}_{,\omega}$ and multiplication by $\tilde{\omega}$ (as defined after Eq. (199)).

Equation (283) then produces the scalar equations

$$(J - \mathcal{M}R^2)\ddot{\gamma}_1 + \mathcal{M}R^2\ddot{\gamma}_2 + (J + \mathcal{M}Rr)\dot{\omega}_1 + \mathcal{M}R^2(\gamma_1 - \gamma_2)(\omega_1^2 + \omega_2^2)$$
$$+ \mathcal{M}Rr\gamma_1(\omega_1^2 + \omega_3^2) + (J + \mathcal{M}Rr)\omega_2\omega_3 - J\gamma_1(\omega_2^2 - \omega_3^2) + k\gamma_1 = 0 \tag{301a}$$

and

$$(J - \mathcal{M}R^2)\ddot{\gamma}_2 + \mathcal{M}R^2\ddot{\gamma}_1 + (J + \mathcal{M}Rr)\dot{\omega}_1 - \mathcal{M}R^2(\gamma_1 - \gamma_2)(\omega_1^2 + \omega_2^2)$$
$$+ 2\mathcal{M}Rr\gamma_2(\omega_1^2 + \omega_3^2) + (J + \mathcal{M}Rr)\omega_2\omega_3 - J\gamma_2(\omega_2^2 - \omega_3^2) + k\gamma_2 = 0 \tag{301b}$$

Equation (269) produces the scalar equations

$$I_1\dot{\omega}_1 + (J + \mathcal{M}Rr)(\ddot{\gamma}_1 + \ddot{\gamma}_2) - I_2\omega_2\omega_3 + (J + \mathcal{M}Rr)(\gamma_1 + \gamma_2)\omega_3^2$$
$$+ I_3\omega_2\omega_3 - (J + \mathcal{M}Rr)(\gamma_1 + \gamma_2)\omega_2^2 = 0 \tag{301c}$$

$$I_2\dot{\omega}_2 - (J + \mathcal{M}Rr)(\dot{\gamma}_1 + \dot{\gamma}_2)\omega_3 - (J + \mathcal{M}Rr)(\gamma_1 + \gamma_2)\dot{\omega}_3$$
$$+ I_1\omega_1\omega_3 + (J + \mathcal{M}Rr)(\dot{\gamma}_1 + \dot{\gamma}_2)\omega_3 - I_3\omega_3\omega_1 - (J + \mathcal{M}Rr)(\gamma_1 + \gamma_2)\omega_1\omega_2 = 0 \tag{301d}$$

$$I_3\dot{\omega}_3 - (J + \mathcal{M}Rr)(\dot{\gamma}_1 + \dot{\gamma}_2)\omega_2 - (J + \mathcal{M}Rr)(\gamma_1 + \gamma_2)\dot{\omega}_2$$
$$- I_1\omega_1\omega_2 - (J + \mathcal{M}Rr)(\dot{\gamma}_1 + \dot{\gamma}_2)\omega_2 + I_2\omega_1\omega_2 - (J + \mathcal{M}Rr)(\gamma_1 + \gamma_2)\omega_1\omega_3 = 0 \tag{301e}$$

After a bit of minor rearranging, these equations become *identical* to those obtained from a Newton-Euler formulation or Kane's approach, and recorded as Eq. (259). Any advantage to be gained one way or another must rest on ease of derivation, which speaks for itself.

It has been noted that one of the valuable features of the methods of analytical mechanics is their generalization of the coordinate concept. In this three-body example, we know that $\eta_1$ and $\eta_2$ are "better" coordinates than $\gamma_1$ and $\gamma_2$ (compare Eqs. (259) and (263)). With Lagrange's method, we should be able to use $\eta_1$ and $\eta_2$ immediately. In practice, however, it may be difficult to recognize the best set of generalized coordinates in advance, or to obtain $T$ in terms of them once they have been chosen. In this case, for example, even the approximate $T$ in Eq. (300) offers little clue that we should substitute

$$\gamma_1 = \frac{1}{2}(\eta_1 + \eta_2)$$

$$\gamma_2 = \frac{1}{2}(\eta_1 - \eta_2)$$

If we recognize the virtues of this substitution on physical grounds, and transform $\overline{T}$ to

$$\overline{T} = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + \frac{1}{4}(J - \mathcal{M}R^2)(\dot{\eta}_1^2 + \dot{\eta}_2^2) + \frac{1}{4}\mathcal{M}R^2(\dot{\eta}_1^2 - \dot{\eta}_2^2)$$

$$+ (J + \mathcal{M}Rr)\,\dot{\eta}_1\omega_1 - \frac{1}{2}\mathcal{M}R^2\eta_2^2(\omega_1^2 + \omega_2^2)$$

$$- \frac{1}{4}\mathcal{M}Rr(\eta_1^2 + \eta_2^2)(\omega_1^2 + \omega_3^2) - \mathcal{M}Rr\eta_1\omega_2\omega_3$$

$$+ \frac{1}{4}J(\eta_1^2 + \eta_2^2)(\omega_2^2 - \omega_3^2) - J\eta_1\omega_2\omega_3$$

then upon application of Eqs. (283) and (284) one does indeed obtain Eqs. (263) directly.

*d. Lagrange's generalized coordinate equations.* To apply the familiar Lagrange equations, as displayed in Eq. (70) of Subsection II-B-1 for example, we must have the kinetic energy $T$ expressed wholly in terms of generalized coordinates, with no quasi-coordinate derivatives permitted. To formulate equations of motion for the three-body system in Fig. 7, we would be obliged to rewrite $\overline{T}$ as it appears in Eq. (299), eliminating $\omega_1$, $\omega_2$, and $\omega_3$ by the substitution of some set of kinematic equations involving three attitude angles and their derivatives; an example can be found in Eq. (170). Linearization in these angles would not be permissible for this problem, and the expression for $T$ would become horribly unwieldy. The differentiations required by Eq. (70) would be straightforward but far more odious than anything we have attempted in this report, and when the job was done the equations would be much more complex in appearance and more difficult to integrate numerically than Eq. (259). It should be obvious that Lagrange's generalized coordinate equations would be a very poor choice for this problem. The first order form of Lagrange's equations proposed in Ref. 35 (see Eqs. (153) and (154) in Subsection C-1) would appear preferable to Lagrange's second order equations, but still not competitive with the other methods considered here.

*e. Hamilton's equations.* If we simply recognize that the interbody springs in the system of Fig. 7 give rise to generalized forces that can be represented as partial derivatives of the scalar potential energy

$$V = \frac{1}{2}k(\gamma_1^2 + \gamma_2^2) \tag{302}$$

then we can apply Hamilton's canonical equations in the form

$$\dot{q} = \mathcal{H}_{,p} \tag{303a}$$

$$\dot{p} = -\mathcal{H}_{,q} \tag{303b}$$

Hamilton's approach suffers the serious disadvantage of requiring the selection of five generalized coordinates for this problem, including three attitude angles for $\mathcal{b}_0$, and then the representation of kinetic energy $T$ in terms of $q$'s and $\dot{q}$'s. As in the case of the Lagrangian generalized coordinate formulation, we would be obliged to rewrite $\bar{T}$ from Eq. (300) as $T(q, \dot{q})$. If we choose to measure the attitude angles of $\mathscr{b}_0$ from an inertially fixed frame, we will find that $T(q, \dot{q})$ is a homogeneous quadratic form in the generalized velocities. This simplifies somewhat the task of calculating the generalized momenta in the matrix $p$, which from Eq. (201) becomes

$$p = T_{,\dot{q}} = M\dot{q} \tag{304}$$

and also simplifies the expression for the Hamiltonian, which from Eq. (156) becomes simply

$$\mathscr{H} = \bar{T}(q, p) + V(q) \tag{305}$$

where $\bar{T}(q, p)$ represents the kinetic energy expressed in terms of generalized coordinates and generalized momenta. (Note that this is not simply $\bar{T}(q, \omega)$ from Eq. (300).)

The major drawback in the Hamiltonian approach is the necessity of inverting $M$ in Eq. (304) and using

$$\dot{q} = M^{-1}p \tag{306}$$

to obtain expressions for $\dot{q}_1, \cdots, \dot{q}_5$ to substitute into $T(q, \dot{q})$ in order to obtain $\bar{T}(q, p)$, and hence $\mathscr{H}$. The matrix $M$ will depend upon the choice of attitude angles for $\mathscr{b}_0$. Inspection of Eq. (300) is sufficient to establish that the 5 by 5 matrix $M$ will be essentially full, and not easily inverted by hand. (If for example $\omega_1$, $\omega_2$, and $\omega_3$ from Eq. (170) were substituted into Eq. (300), only four of the twenty-five terms in $M$ would be zeros.)

Even without seeing the results, we might speculate about the structure of Hamilton's equations for the three-body system of Fig. 7. We would have ten first order equations (rather than effectively seven first order equations appearing in Eq. (259)). Hamilton's equations would involve attitude angles in a way that will probably result in lengthy expressions for the right side of Eq. (303). But we are certain that the left side of Eq. (303) is a single column matrix of first derivatives, with no time-varying coefficient matrix. Because the repeated inversion (or Gaussian elimination processing) of the coefficient matrix is a serious computational burden, Hamilton's equations would be attractive for digital computer numerical integration once they finally were obtained. It would be sophistry to argue however that these facts establish an advantage of Hamilton's equations over any of the methods that lead to Eq. (259); if we are willing to invert a 5 by 5 matrix by hand to obtain Hamilton's equations, then we can also invert the 5 by 5 coefficient matrix of highest derivatives in the matrix version of Eq. (259), and again circumvent the numerical problems associated with a variable coefficient matrix.

*f. Summary for the symmetric three-body example.* Equations of motion have been derived for the small deformation and unrestricted rotation of the system in Fig. 7 in four distinct ways, although only two derivations have been presented in detail in this report. The quasi-coordinate formulations of Kane and of Lagrange have been applied explicitly here. In Appendix C of Ref. 31, a computer algorithm for equation assembly is applied to this same example; this algorithm has its roots in the methods of Newton and Euler, but the Newton–Euler equations have been manipulated to eliminate constraint forces and torques in a way proposed by Hooker (Ref. 26). Finally, in an ad hoc scratch sheet derivation not recorded in print, the author has derived equations of motion for this system by means of first principles of Newton and Euler.

Although these four methods appear to be very different from one another, they all produced the same equations of motion, recorded here as Eq. (259).

Preliminary investigation was enough to justify the conclusion that the application of Lagrange's generalized coordinate equations would produce equations of motion that were different from and far more complex than those obtained by the four methods previously discussed.

Attempts to apply Hamilton's canonical equations were thwarted by the necessity of inverting a full 5 by 5 matrix by hand. This approach is clearly unacceptable for most multiple-rigid-body models of spacecraft.

**4. Point-connected rigid bodies in a topological tree.** Experience with previous examples generates the suspicion that some of the different equation derivation methods in the literature are less different than they appear; for the holonomic systems considered thus far in this report there have emerged identical sets of equations of motion from several quite different derivation procedures. In this section we will establish that three different methods give identical equations for a class of mathematical models of considerable theoretical interest and great practical utility: the system of point-connected rigid bodies in a topological tree.

A set of $n + 1$ rigid bodies all interconnected by $n$ points, each of which is common to two bodies, has been labeled in the literature as *a system of point-connected rigid bodies in a topological tree*. The tree topology, which infers the absence of closed loops formed by chains of rigid bodies, is implied by the presence of connections that number one less than the number of bodies, as long as *all* bodies are interconnected. With this topology there is a unique internal path between any two bodies of the system. Without loss of generality, one can conceive of any set of point-connected rigid bodies as a (possibly larger) set of rigid bodies interconnected by *line hinges*, admitting the possibility that some bodies might be massless and dimensionless abstractions.

The problem of equation formulation for multiple-rigid-body systems has a rich modern literature, because spacecraft simulation requirements made the problem important at the same time that digital computer development made it possible to extract information from the equations by means of numerical integrations. The contributions of Hooker, Margulies, Roberson, Wittenburg, Velman, and Russell have been noted previously in this report (see References). Early efforts were plagued by the presence of interbody constraint torques in the equations (Refs. 13–15 for example), but in the methods of Russell (Ref. 18) and Hooker (Ref. 26) these torques are eliminated and the equations are reduced to their minimum possible dimensions. Because Russell's contribution is not a set of explicit equations of motion of $n + 1$ bodies but rather a method for generating such equations and a computer program for processing them, we cannot without digression compare his results to those obtained by the procedures of analytical dynamics that we are examining in this report.[11] Hooker's equations, in contrast, are fully explicit, and we can readily compare his results to those developed in this report. Hooker's method, which is a variation of the earlier method of Hooker and Margulies (Ref. 13), has been adopted as the foundation for both analytical and computational work at JPL, and there is available in Ref. 31 a more detailed exposition of Hooker's ideas than can be found in Hooker's own original brief paper (Ref. 26). In what follows, therefore, Ref. 31 is adopted as the source for comparisons with the "Hooker–Margulies/Hooker" equations. In particular, this means that the labeling convention of Ref. 31 is retained in this report, and the practice of working exclusively with line hinge connections is adopted.

In what follows, the equations of motion for a point-connected set of rigid bodies in a topological tree are briefly presented as they appear in Ref. 31, and then the quasi-coordinate methods of Kane and of Lagrange are applied to the same system. The final equations are shown to be identical. Alternative derivation procedures (such as Hamilton's) are discussed briefly.

*a. The Hooker–Margulies/Hooker equations.* The equations of motion of $n + 1$ rigid bodies $\mathscr{b}_0, \mathscr{b}_1, \cdots, \mathscr{b}_n$ interconnected by $n$ line hinges are, from Eq. (1) of Ref. 31, given by

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0n} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1n} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n0} & a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \dot{\omega}^0 \\ \ddot{\gamma}_1 \\ \ddot{\gamma}_2 \\ \cdot \\ \cdot \\ \cdot \\ \ddot{\gamma}_n \end{bmatrix} = \begin{bmatrix} \sum\limits_{k \epsilon \mathscr{B}} C^{0k} A^k \\ g^{1^T} \sum\limits_{k \epsilon \mathscr{P}} \varepsilon_{1k} C^{1k} A^k + \tau_1 \\ g^{2^T} \sum\limits_{k \epsilon \mathscr{P}} \varepsilon_{2k} C^{2k} A^k + \tau_2 \\ \cdot \\ \cdot \\ \cdot \\ g^{n^T} \sum\limits_{k \epsilon \mathscr{P}} \varepsilon_{nk} C^{nk} A^k + \tau_n \end{bmatrix} \tag{307}$$

Here $\omega^0$ is the 3 by 1 matrix of the inertial angular velocity of $\mathscr{b}_0$ for a vector basis $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$ fixed in $\mathscr{b}_0$. The angle $\gamma_1$ describes the rotation of $\mathscr{b}_1$ about a hinge line common to $\mathscr{b}_1$ and $\mathscr{b}_0$, with positive sense established by a unit vector $\mathbf{g}^1$. Similarly $\gamma_2$ describes the rotation of $\mathscr{b}_2$ relative to a contiguous body of lesser index, and so on up to $\gamma_n$. The coefficient matrix on the left side of Eq. (307) is symmetric, and generally depends on time either explicitly or through the angles $\gamma_1, \cdots, \gamma_n$. Because we will not dwell upon Eq. (307) in this report (preferring to concentrate on the antecedents in the derivation of this equation), we can skip the lengthy process of explicitly defining the elements of the left side coefficient matrix here; Ref. 31 provides these definitions on page 14.[12] The symbol $A^k$ is also not of specific interest here; it includes external forces and torques and a large collection of terms involving system parameters and kinematic variables having lower order derivatives than appear explicitly on the left side of Eq. (307). See p. 15 of Ref. 31 for the definition of $A^k$. We are concerned with the remaining symbols on the

[11]Russell's method warrants the comparative evaluation accorded here to other methods, but the present report is limited to methods of analytical dynamics.

right side of Eq. (307), which are defined as follows (see pages 10–15 of Ref. 31 for more details):

(1) $\mathcal{B} \triangleq \{0, 1, \cdots, n\}$.

(2) $\mathcal{P} \triangleq \{1, 2, \cdots, n\}$.

(3) $C^{jk}$ is a direction cosine matrix relating the orientations of $\mathscr{b}_j$ and $\mathscr{b}_k$.

(4) $\tau_k$ is the component along the hinge axis of the torque applied to $\mathscr{b}_k$ by the attached body of lesser index.

(5) $g^k$ is the 3 by 1 matrix representing $\mathbf{g}^k$ in a vector basis fixed in $\mathscr{b}_k$.

(6) $\varepsilon_{jk}$ is a *path element* such that $\varepsilon_{jk} \triangleq \begin{cases} 1 \text{ if hinge } j \text{ lies between } \mathscr{b}_0 \text{ and } \mathscr{b}_k. \\ 0 \text{ otherwise} \end{cases}$

**Equation (307) is important here only because it provides an indication of the structure of the equations of motion as they are processed by the digital computer for numerical integration. More important for the analytical purposes at hand is a version of Eq. (307) that appears in Appendix A of Ref. 31, part way through the derivation of that final result[13]. Equations (A.14) and (A.15) on page 78 of Ref. 31 appear after minor notational revision and condensation as**

$$\sum_{j\in\mathcal{B}} [\mathbf{T}^j + \sum_{s\in\mathcal{B}} (\mathbf{D}^{js} \times \mathbf{f}^s - \mathbf{L}^{js} \times \mathcal{M}_s \ddot{\mathbf{p}}^s) - \dot{\mathbf{H}}^j] = 0$$

**or, after exchanging indices in the double sum terms, dot-multiplying by $\mathbf{b}_i^0$, and reversing the summation sequence,**

$$\mathbf{b}_i^0 \cdot \sum_{j\in\mathcal{B}} [\mathbf{T}^j - \dot{\mathbf{H}}^{c_j} + \sum_{s\in\mathcal{B}} (\mathbf{D}^{sj} \times \mathbf{f}^j - \mathbf{L}^{sj} \times \mathcal{M}_j \ddot{\mathbf{p}}^j)] = 0, \qquad i = 1, 2, 3 \tag{308a}$$

and also (from Eq. (A.15))

$$\tau_k + \mathbf{g}^k \cdot \sum_{j\in\mathcal{P}} \varepsilon_{kj} [\mathbf{T}^j + \sum_{s\in\mathcal{B}} (\mathbf{D}^{js} \times \mathbf{f}^s - \mathbf{L}^{js} \times \mathcal{M}_s \ddot{\mathbf{p}}^s) - \dot{\mathbf{H}}^j] = 0 \qquad (s \in \mathcal{P}) \tag{308b}$$

We shall refer to Eq. (308) in what follows as the preliminary form of the Hooker–Margulies/Hooker equations. As elsewhere in this report, $\mathbf{H}_j$ is the inertial angular momentum of rigid body $\mathscr{b}_k$ with respect to its mass center $c_j$, $\mathcal{M}_s$ is the mass of $\mathscr{b}_s$, and $\mathbf{f}^s$ is the force applied to $\mathscr{b}_s$ *exclusive of nonworking constraint forces*. The symbol $\mathbf{T}^j$ represents any external torques applied to $\mathscr{b}_j$, excluding torques due to contact with other bodies of the system. The symbol $\boldsymbol{\rho}^s$ represents the position vector of the mass center $c_s$ of $\mathscr{b}_s$ with respect to the system mass center (CM).

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[12]Definition 39 on page 14 of Ref. 31 should read

$$a_{0k} \triangleq \sum_{r\in\mathcal{B}} \sum_{j\in\mathcal{B}} \varepsilon_{kj} C^{0j} \Phi^{rj} C^{rk} g^k = \sum_{r\in\mathcal{P}} \varepsilon_{kr} C^{0r} \Phi^{rr} C^{rk} g^k$$
$$- \mathcal{M} \sum_{r\in\mathcal{B}} \sum_{j\in\mathcal{B}-r} \varepsilon_{kj} (C^{0r} D^{jr T} C^{jr} D^{rj} - C^{0j} D^{jr} D^{jr T}) C^{rk} g^k$$

[13]As shown in Ref. 31, the path from Eqs. (308) to Eq. (307) is an arduous one, involving the construction of difficult kinematical relationships that represent an important part of the contribution of the original paper by Hooker and Margulies (Ref. 13).

The position vectors $\mathbf{D}^{js}$ and $\mathbf{L}^{js}$ in Eq. (308) are as defined in Ref. 31, page 13, with the following implications.

The position vector from $c_k$ to the hinge point on $\mathcal{b}_k$ leading to $\mathcal{b}_r$ is called $\mathbf{L}^{kr}$, with $\mathbf{L}^{kk} \triangleq 0$. A point $b_k$ (called the barycenter of $\mathcal{b}_k$) is located with respect to $c_k$ by the position vector

$$\mathbf{D}^{kk} \triangleq -\sum_{j \in \mathcal{B}} \mathbf{L}^{kj} \mathcal{M}_j / \mathcal{M} \tag{309a}$$

and $\mathbf{D}^{kr}$ is defined by

$$\mathbf{D}^{kr} \triangleq \mathbf{D}^{kk} + \mathbf{L}^{kr} \tag{309b}$$

As shown in Appendix A, Eq. (A-22), of Ref. 31,

$$\boldsymbol{\rho}^r = \sum_{s \in \mathcal{B}} \mathbf{D}^{sr} \tag{310}$$

(This identity, first established by Hooker and Margulies, is far from obvious, and is an important element of Ref. 13.)

The immediate objective is to apply first Kane's quasi-coordinate equations and then Lagrange's quasi-coordinate and generalized coordinate equations to the set of point-connected rigid bodies in a topological tree, in order to compare the results with Eq. (308).

*b. Kane's quasi-coordinate equations.* As developed in Subsection II-A-4, Kane's quasi-coordinate formulation of D'Alembert's principle can be applied to $n + 1$ point-connected rigid bodies interconnected by $n$ line hinges by selecting the quasi-coordinate derivatives

$$u_j \triangleq \dot{\gamma}_j, \qquad j = 1, \cdots, n \tag{311a}$$

$$u_{j+n} \triangleq \omega_j^0 \triangleq \boldsymbol{\omega}^0 \cdot \mathbf{b}_j^0, \qquad j = 1, 2, 3 \tag{311b}$$

and recording the equations of motion (see Eq. (66a))

$$f_k + f_k^* = 0, \qquad k = 1, 2, \cdots, n+3 \tag{312}$$

where (as in Eqs. (65c) and (65d))

$$f_k \triangleq \sum_{j \in \mathcal{B}} [\mathbf{f}^j \cdot \mathbf{V}_k^{c_j} + \mathbf{m}^j \cdot \boldsymbol{\omega}_k^j], \qquad k = 1, 2, \cdots, n+3 \tag{313a}$$

and

$$f_k^* \triangleq -\sum_{j \in \mathcal{B}} [\mathcal{M}_j \ddot{\mathbf{R}}^j \cdot \mathbf{V}_k^{c_j} + \dot{\mathbf{H}}^j \cdot \boldsymbol{\omega}_k^j], \qquad k = 1, 2, \cdots, n+3 \tag{313b}$$

The vectors $\mathbf{V}^{c_j}_k$ and $\boldsymbol{\omega}^j_k$ are obtained from the definitions

$$\dot{\mathbf{R}}^j \stackrel{\Delta}{=} \sum_{k=1}^{n+3} \mathbf{V}^{c_j}_k u_k + \mathbf{V}^{c_j}_t \tag{314a}$$

and

$$\boldsymbol{\omega}^j \stackrel{\Delta}{=} \sum_{k=1}^{n+3} \boldsymbol{\omega}^j_k u_k + \boldsymbol{\omega}_t \tag{314b}$$

where $\dot{\mathbf{R}}^j$ and $\boldsymbol{\omega}^j$ are respectively the mass center inertial velocity and the inertial angular velocity of the $j$th body.

For this example, as noted in Eq. (A-38a) of Ref. 31,

$$\boldsymbol{\omega}^j = \boldsymbol{\omega}^0 + \sum_{r \epsilon \mathcal{P}} \varepsilon_{rj} \dot{\gamma}_r \mathbf{g}^r \tag{315}$$

Comparison of Eqs. (314b) and (315) in the light of Eqs. (311) yields half of Kane's coefficient vectors, namely

$$\boldsymbol{\omega}^j_k = \varepsilon_{kj} \mathbf{g}^k, \qquad j \epsilon \mathcal{B} \text{ and } k \epsilon \mathcal{P} \tag{316a}$$

and

$$\boldsymbol{\omega}^j_{n+i} = \mathbf{b}^0_i, \qquad j \epsilon \mathcal{B} \text{ and } i = 1, 2, 3 \tag{316b}$$

To obtain the remaining coefficient vectors, we require the kinematic expansion of $\dot{\mathbf{R}}^j$ for comparison with Eq. (314a). If $\mathbf{R}^c$ is the inertial position vector of the system mass center, then we have

$$\mathbf{R}^j = \mathbf{R}^c + \boldsymbol{\rho}^j \tag{317}$$

or, with Eq. (310),

$$\mathbf{R}^j = \mathbf{R}^c + \sum_{s \epsilon \mathcal{B}} \mathbf{D}^{sj} \tag{318}$$

Since $\mathbf{D}^{sj}$ is a vector fixed in $\mathscr{b}_s$, we can write

$$\dot{\mathbf{R}}^j = \dot{\mathbf{R}}^c + \sum_{s \epsilon \mathcal{B}} \dot{\mathbf{D}}^{sj} = \dot{\mathbf{R}}^c + \sum_{s \epsilon \mathcal{B}} \boldsymbol{\omega}^s \times \mathbf{D}^{sj} \tag{319}$$

Now the combination of Eqs. (319), (315), and (311) gives

$$\begin{aligned} \dot{\mathbf{R}}^j &= \dot{\mathbf{R}}^c + \sum_{s \epsilon \mathcal{B}} \left[ \boldsymbol{\omega}^0 + \sum_{r \epsilon \mathcal{P}} \varepsilon_{rs} \dot{\gamma}_r \mathbf{g}^r \right] \times \mathbf{D}^{sj} \\ &= \dot{\mathbf{R}}^c + \sum_{s \epsilon \mathcal{B}} \left[ u_{n+1} \mathbf{b}^0_1 + u_{n+2} \mathbf{b}^0_2 + u_{n+3} \mathbf{b}^0_3 + \sum_{r \epsilon \mathcal{P}} \varepsilon_{rs} u_r \mathbf{g}^r \right] \times \mathbf{D}^{sj} \end{aligned} \tag{320}$$

This result, when compared to Eq. (314a) provides

$$\mathbf{V}^{c_j}_{k} = \mathbf{g}^k \times \sum_{s\epsilon\mathcal{B}} \varepsilon_{ks}\mathbf{D}^{sj}, \qquad j\epsilon\mathcal{B} \text{ and } k\epsilon\mathcal{P} \tag{321a}$$

and

$$\mathbf{V}^{c_j}_{n+i} = \mathbf{b}^0_i \times \sum_{s\epsilon\mathcal{B}} \mathbf{D}^{sj}, \qquad j\epsilon\mathcal{B} \text{ and } i = 1, 2, 3 \tag{321b}$$

The combination of Eqs. (312), (313), (316b), and (321b) produces the three scalar equations of motion

$$\sum_{j\epsilon\mathcal{B}} \left[ (\mathbf{f}^j - \mathcal{M}_j\ddot{\mathbf{R}}^j) \cdot \left( \mathbf{b}^0_i \times \sum_{s\epsilon\mathcal{B}} \mathbf{D}^{sj} \right) + (\mathbf{m}^j - \dot{\mathbf{H}}^j) \cdot \mathbf{b}^0_i \right] = 0, \qquad i = 1, 2, 3 \tag{322a}$$

The combination of Eqs. (312), (313), (316a), and (321a) produces the $n$ scalar equations of motion

$$\sum_{j\epsilon\mathcal{B}} \left[ \mathbf{f}^j - \mathcal{M}_j\ddot{\mathbf{R}}^j) \cdot \left( \mathbf{g}^k \times \sum_{s\epsilon\mathcal{B}} \varepsilon_{ks}\mathbf{D}^{sj} \right) + (\mathbf{m}^j - \dot{\mathbf{H}}^j) \cdot \varepsilon_{kj}\mathbf{g}^k \right] = 0, \qquad k = 1, \cdots, n \tag{322b}$$

With the vector identity

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{c} \cdot \mathbf{a} = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$$

for any vetors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, Eqs. (322) become

$$\mathbf{b}^0_i \cdot \sum_{j\epsilon\mathcal{B}} \left\{ \sum_{s\epsilon\mathcal{B}} \mathbf{D}^{sj} \times (\mathbf{f}^j - \mathcal{M}_j\ddot{\mathbf{R}}^j) + (\mathbf{m}^j - \dot{\mathbf{H}}^j) \right\} = 0, \qquad i = 1, 2, 3 \tag{323a}$$

and

$$\sum_{j\epsilon\mathcal{B}} \left\{ \mathbf{g}^k \cdot \left[ \sum_{s\epsilon\mathcal{B}} \varepsilon_{ks}\mathbf{D}^{sj} \times (\mathbf{f}^j - \mathcal{M}_j\ddot{\mathbf{R}}^j) + \varepsilon_{kj}(\mathbf{m}^j - \dot{\mathbf{H}}^j) \right] \right\} = 0, \qquad k = 1, \cdots, n \tag{323b}$$

Motivated by past examples, we adopt the hypotheses that Eq. (323a) is identical to Eq. (308a), and that Eq. (323b) is identical to Eq. (308b). In comparing these equations, we must recognize that $\mathbf{m}^j$ and $\mathbf{T}^j$ differ only in that $\mathbf{m}^j = \mathbf{T}^j$ plus any moments about $c_j$ due to interaction torques other than nonworking constraint torques applied to $\mathcal{b}_j$ by contiguous bodies. In particular, interaction torques such as $\tau_j\mathbf{g}^j$ would have to be added to $\mathbf{T}^j$ to obtain $\mathbf{m}^j$. In Eq. (323a), however, the summation over $j\epsilon\mathcal{B}$ produces these interaction torques only in equal and opposite pairs, so that

$$\sum_{j\epsilon\mathcal{B}} \mathbf{m}^j = \sum_{j\epsilon\mathcal{B}} \mathbf{T}^j \tag{324}$$

With Eq. (324), it becomes apparent that Eqs. (323a) and Eq. (308a) are identical if and only if the quantity

$$\sum_{j\epsilon\mathcal{B}}\sum_{s\epsilon\mathcal{B}}(\mathbf{L}^{sj}\times\mathcal{M}_j\ddot{\boldsymbol{\rho}}^j-\mathbf{D}^{sj}\times\mathcal{M}_j\ddot{\mathbf{R}}^j)$$

is zero. With the substitution of Eqs. (309) for $\mathbf{D}^{sj}$ and Eq. (317) for $\mathbf{R}^j$, this expression becomes

$$\sum_{j\epsilon\mathcal{B}}\sum_{s\epsilon\mathcal{B}}\left\{\mathbf{L}^{sj}\times\mathcal{M}_j\ddot{\boldsymbol{\rho}}^j-\left(-\sum_{r\epsilon\mathcal{B}}\mathbf{L}^{sr}\frac{\mathcal{M}_r}{\mathcal{M}}+\mathbf{L}^{sj}\right)\times\mathcal{M}_j(\ddot{\mathbf{R}}^c+\ddot{\boldsymbol{\rho}}^j)\right\}=$$

$$\sum_{j\epsilon\mathcal{B}}\sum_{s\epsilon\mathcal{B}}\left\{-\mathbf{L}^{sj}\times\mathcal{M}_j\left(\frac{\mathbf{F}}{\mathcal{M}}\right)+\sum_{r\epsilon\mathcal{B}}\mathbf{L}^{sr}\frac{\mathcal{M}_r}{\mathcal{M}}\times\mathcal{M}_j\left(\frac{\mathbf{F}}{\mathcal{M}}+\ddot{\boldsymbol{\rho}}^j\right)\right\} \tag{325}$$

where the resultant force $\mathbf{F}$ on the total system obeys Newton's second law in the form

$$\mathbf{F}=\mathcal{M}\ddot{\mathbf{R}}^c \tag{326}$$

By mass center definition

$$\sum_{j\epsilon\mathcal{B}}\mathcal{M}_j\boldsymbol{\rho}^j=0 \tag{327}$$

so the terms in Eq. (325) involving $\ddot{\boldsymbol{\rho}}^j$ sum to zero, and Eq. (325) becomes (noting Eqs. (309))

$$-\sum_{j\epsilon\mathcal{B}}\sum_{s\epsilon\mathcal{B}}\left[\left(\mathbf{L}^{sj}-\sum_{r\epsilon\mathcal{B}}\mathbf{L}^{sr}\mathcal{M}_r/\mathcal{M}\right)\times\mathcal{M}_j\mathbf{F}/\mathcal{M}\right]=-\sum_{j\epsilon\mathcal{B}}\sum_{s\epsilon\mathcal{B}}\mathcal{M}_j\mathbf{D}^{sj}\times\mathbf{F}/\mathcal{M}=0 \tag{328}$$

since by Eq. (A-25) of Ref. 31

$$\sum_{s\epsilon\mathcal{B}}\mathcal{M}_j\mathbf{D}^{sj}=0 \tag{329}$$

Thus the equivalence of the three scalar equations in Eqs. (323a) and the vector equation (308a) is established.

In comparing Eqs. (323b) and (308b), one must first recognize that $\epsilon_{k0}=0$ for all $k$, so that the apparent difference in summation ranges for $j$ in these two equations has no significance. Proper interpretation of $\mathbf{T}^j$ and $\mathbf{m}^j$ produces the identity

$$\tau_k+\mathbf{g}^k\cdot\sum_{j\epsilon\mathcal{P}}\epsilon_{kj}\mathbf{T}^j=\sum_{j\epsilon\mathcal{B}}\epsilon_{kj}\mathbf{g}^k\cdot\mathbf{m}^j \tag{330}$$

since the path symbols $\epsilon_{kj}$ have the effect of limiting the range of the summation to those bodies $\mathscr{b}_j$ for which $\mathscr{b}_k$ is on the path between $\mathscr{b}_j$ and $\mathscr{b}_0$; thus all of the interbody torques that appear in $\mathbf{m}^j$ (but not in $\mathbf{T}^j$) are summed in equal and opposite pairs, with the single exception of $\tau_k\mathbf{g}^k$.

Thus for Eqs. (308b) and (323b) to be equal, we require only that the expression

$$\mathbf{g}^k \cdot \sum_{j\epsilon\mathcal{B}} \varepsilon_{kj} \sum_{s\epsilon\mathcal{B}} (\mathbf{D}^{js} \times \mathbf{f}^s - \mathbf{L}^{js} \times \mathcal{M}_s \ddot{\boldsymbol{\rho}}^s) - \sum_{j\epsilon\mathcal{B}} \mathbf{g}^k \cdot \sum_{s\epsilon\mathcal{B}} \varepsilon_{ks} \mathbf{D}^{sj} \times (\mathbf{f}^j - \mathcal{M}_j \ddot{\mathbf{R}}^j)$$

reduce to zero. The identity

$$\mathbf{g}^k \cdot \sum_{j\epsilon\mathcal{B}} \varepsilon_{kj} \sum_{s\epsilon\mathcal{B}} \mathbf{D}^{js} \times \mathbf{f}^s = \mathbf{g}^k \cdot \sum_{s\epsilon\mathcal{B}} \varepsilon_{ks} \sum_{j\epsilon\mathcal{B}} \mathbf{D}^{sj} \times \mathbf{f}^j$$

eliminates all forces in the expression of interest, which then becomes (exchanging dummy indices $j$ and $s$ in the first double sum, and reversing the summation sequence)

$$-\mathbf{g}^k \cdot \sum_{j\epsilon\mathcal{B}} \varepsilon_{kj} \sum_{s\epsilon\mathcal{B}} \mathbf{L}^{js} \times \mathcal{M}_s \ddot{\boldsymbol{\rho}}^s + \sum_{j\epsilon\mathcal{B}} \mathbf{g}^k \cdot \sum_{s\epsilon\mathcal{B}} \varepsilon_{ks} \mathbf{D}^{sj} \times \mathcal{M}_j \ddot{\mathbf{R}}^j =$$

$$\mathbf{g}^k \cdot \sum_{j\epsilon\mathcal{B}} \sum_{s\epsilon\mathcal{B}} \varepsilon_{ks} (\mathbf{D}^{sj} \times \mathcal{M}_j \ddot{\mathbf{R}}^j - \mathbf{L}^{sj} \times \mathcal{M}_j \ddot{\boldsymbol{\rho}}^j) \tag{331}$$

We can again use Eqs. (309) for $\mathbf{D}^{sj}$ and Eqs. (317) and (326) for $\mathbf{R}^j$, and rewrite the parenthetical term in Eq. (331) as

$$\mathbf{D}^{sj} \times \mathcal{M}_j \ddot{\mathbf{R}}^j - \mathbf{L}^{sj} \times \mathcal{M}_j \ddot{\boldsymbol{\rho}}^j = \left( - \sum_{r\epsilon\mathcal{B}} \mathbf{L}^{sr} \frac{\mathcal{M}_r}{\mathcal{M}} + \mathbf{L}^{sj} \right) \times \mathcal{M}_j \left( \frac{\mathbf{F}}{\mathcal{M}} + \ddot{\boldsymbol{\rho}}^j \right)$$

$$- \mathbf{L}^{sj} \times \mathcal{M}_j \ddot{\boldsymbol{\rho}}^j = \left( - \sum_{r\epsilon\mathcal{B}} \mathbf{L}^{sr} \frac{\mathcal{M}_r}{\mathcal{M}} \right) \times \mathcal{M}_j \ddot{\boldsymbol{\rho}}^j + \mathcal{M}_j \mathbf{D}^{sj} \times \frac{\mathbf{F}}{\mathcal{M}} \tag{332}$$

In returning this expression to Eq. (331), one finds that the terms in Eq. (332) involving $\mathcal{M}_j \ddot{\boldsymbol{\rho}}^j$ sum to zero by virtue of Eq. (327), and those involving $\mathcal{M}_j \mathbf{D}^{sj}$ sum to zero by virtue of Eq. (329). Thus the proposition is proven, the expression in Eq. (331) is zero, and Eqs. (323b) and (308b) are identical.

*c. Lagrange's equations.* As shown in Subsection B-1, Lagrange's equations in terms of independent generalized coordinates provide exactly the same results as may be obtained from Lagrange's form of D'Alembert's principle; the differences in these two methods involve only the operational procedures employed to achieve these equations. Moreover, the same equations emerge from Kane's quasi-coordinate form of D'Alembert's principle whenever the "quasi-coordinates" are actually generalized coordinates; in this special case Kane's D'Alembert method is identical to that of Lagrange, as shown in Subsection A-4.

In application to the multiple-rigid-body problem, with quasi-coordinate derivatives selected as in Eqs. (311), one can therefore conclude in advance of derivation that $n$ of the $n + 3$ scalar equations of motion obtained by Kane's approach would also emerge (after many hours of labor) from a Lagrangian generalized coordinate formulation; these are the equations represented here by Eq. (323b), or (equivalently) Eq. (308b).

The three remaining equations obtained by Kane's method (or by the Hooker–Margulies/Hooker procedure) most certainly cannot be obtained by a Lagrangian generalized coordinate approach, since the kinematic variables $\omega_1^0, \omega_2^0, \omega_3^0$ are not derivatives of generalized coordinates. It will be shown here, however, that these three equations can be obtained by the application of Lagrange's quasi-coordinate equations, as represented by Eq. (284).

In application to a holonomic system, Eq. (284) becomes

$$\frac{d}{dt}(\overline{T}_{,\omega}) + \tilde{\omega}\overline{T}_{,\omega} - W_0^{-1}\overline{T}_{,q^0} = W_0^{-1}Q^0 \tag{333}$$

For the multiple-rigid-body problem, $\omega$ is the 3 by 1 matrix representing $\boldsymbol{\omega}^0$ in vector basis $\underline{\mathbf{b}}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$. The kinetic energy expression in terms of $\omega, \gamma_1, \cdots, \gamma_n, \dot{\gamma}_1, \cdots, \dot{\gamma}_n$ is called $\overline{T}$. The 3 by 1 matrix $q_0$ contains some set of generalized coordinates (such as Euler angles) defining the inertial orientation of $\mathscr{b}_0$. These are irrelevant to the physical problem, and will not appear in $\overline{T}$, so that $\overline{T}_{,q^0}$ will be zero. The matrix $W_0$ is defined by the combination of Eqs. (280) and (134d), such that

$$\mathbf{W}_0 = \omega^T_{,\dot{q}^0} \tag{334}$$

The 3 by 1 matrix of generalized forces $Q^0$ has the elements (from Eq. (22))

$$Q_k^0 = \sum_{j\in\mathscr{B}}\left(\frac{\partial\dot{\mathbf{R}}^j}{\partial\dot{q}_k^0}\cdot\mathbf{f}^j + \frac{\partial\boldsymbol{\omega}^j}{\partial\dot{q}_k^0}\cdot\mathbf{m}^j\right)$$

which with the substitution of Eq. (319) for $\dot{\mathbf{R}}^j$ becomes

$$Q_k^0 = \sum_{j\in\mathscr{B}}\left[\left(\sum_{s\in\mathscr{B}}\frac{\partial\boldsymbol{\omega}^s}{\partial\dot{q}_k^0}\times\tilde{\mathbf{D}}^{sj}\right)\cdot\mathbf{f}^j + \frac{\partial\boldsymbol{\omega}^j}{\partial\dot{q}_k^0}\cdot\mathbf{m}^j\right] = \sum_{j\in\mathscr{B}}\left[\sum_{s\in\mathscr{B}}\frac{\partial\boldsymbol{\omega}^s}{\partial\dot{q}_k^0}\cdot(\mathbf{D}^{sj}\times\mathbf{f}^j) + \frac{\partial\boldsymbol{\omega}^j}{\partial\dot{q}_k^0}\cdot\mathbf{m}^j\right]$$

Eq. (315) reveals that, for any index $r$,

$$\frac{\partial\boldsymbol{\omega}^r}{\partial\dot{q}_k^0} = \frac{\partial\boldsymbol{\omega}^0}{\partial\dot{q}_k^0}$$

so that the generalized force becomes

$$Q_k^0 = \frac{\partial\boldsymbol{\omega}^0}{\partial\dot{q}_k^0}\cdot\sum_{j\in\mathscr{B}}\left[\sum_{s\in\mathscr{B}}\mathbf{D}^{sj}\times\mathbf{f}^j + \mathbf{m}^j\right], \qquad k = 1, 2, 3 \tag{335}$$

The matrix counterpart to the three equations implied by Eq. (335) is (with all vectors in basis $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$)

$$Q^0 = \omega^T_{,\dot{q}^0}\sum_{j\in\mathscr{B}}\left(\sum_{s\in\mathscr{B}}\tilde{D}^{sj}f^j + m^j\right) \tag{336}$$

so that in Eq. (333) the term $W_0^{-1}Q^0$ becomes

$$\begin{aligned} W_0^{-1}Q^0 &= (\omega^T_{,\dot{q}_0})^{-1}\,\omega^T_{,\dot{q}^0}\sum_{j\in\mathscr{B}}\left(\sum_{s\in\mathscr{B}}\tilde{D}^{sj}f^j + m^j\right) \\ &= \sum_{s\in\mathscr{B}}\left(\sum_{j\in\mathscr{B}}\tilde{D}^{sj}f^j + m^j\right) \end{aligned} \tag{337}$$

Thus the term $W_0^{-1}Q^0$ in Eq. (333) corresponds to those terms in Eq. (323a) involving $\mathbf{f}^j$ and $\mathbf{m}^j$; more specifically

$$\sum_{j\in\mathcal{B}} \left( \sum_{s\in\mathcal{B}} \mathbf{D}^{sj} \times \mathbf{f}^j + \mathbf{m}^j \right) = \{\mathbf{b}_1^0 \mathbf{b}_2^0 \mathbf{b}_3^0\} \sum_{j\in\mathcal{B}} \left( \sum_{s\in\mathcal{B}} \tilde{D}^{sj} f^j + m^j \right)$$

and the scalar terms in the three rows of the column matrix in Eq. (337) are identical to those produced by the dot-multiplications of $\mathbf{b}_i^0$ with the terms involving $\mathbf{f}^j$ and $\mathbf{m}^j$, for $i = 1, 2, 3$, in Eq. (323a).

Moreover, we can now recognize that the elements of the matrix $Q^0$ are no more than the scalar components in vector basis $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$ of the external moment applied to the multiple-rigid-body system about the system mass center. This identification requires only the substitution from Eq. (310) of

$$\boldsymbol{\rho}^j = \sum_{s\in\mathcal{B}} \mathbf{D}^{sj}$$

into Eq. (335) or its successors, and the recognition that (in view of the third law cancellation of interbody forces)

$$\mathbf{M} = \sum_{j\in\mathcal{B}} (\boldsymbol{\rho}^j \times \mathbf{f}^j + \mathbf{m}^j) \tag{338}$$

where $\mathbf{M}$ represents the external moment applied to the system about the system mass center.

There remains for the proof of the exact equivalence of Kane's equation (323a) and Lagrange's equation (333) only the identification of the terms $d/dt\,(\bar{T}_{,\omega}) + \tilde{\omega}\bar{T}_{,\omega}$ in Eq. (333) with the terms involving $\dot{\mathbf{H}}^j$ and $\ddot{\mathbf{R}}^j$ in Eq. (323a), for $i = 1, 2, 3$. This identification is most readily accomplished by recognizing that (by virtue of Eq. (338)) these terms must also be identical to the expression in vector basis $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$ of the inertial time derivative of the angular momentum $\mathbf{H}$ of the total system with respect to the system mass center. In other words, Eqs. (333) and (323a) must both be identical to

$$\mathbf{b}_i^0 \cdot (\mathbf{M} - \dot{\mathbf{H}}) = 0 \qquad i = 1, 2, 3 \tag{339}$$

The identity of Eqs. (323a) and (339) is made obvious by the expression

$$\mathbf{H} = \sum_{j\in\mathcal{B}} (\mathbf{H}^j + \mathcal{M}_j \boldsymbol{\rho}_j \times \dot{\boldsymbol{\rho}}_j) \tag{340}$$

and its derivative (noting Eqs. (310) and (327))

$$\dot{\mathbf{H}} = \sum_{j\in\mathcal{B}} (\dot{\mathbf{H}}^j + \mathcal{M}_j \boldsymbol{\rho}_j \times \ddot{\boldsymbol{\rho}}_j) = \sum_{j\in\mathcal{B}} \left[ \dot{\mathbf{H}}^j + \sum_{s\in\mathcal{B}} \mathbf{D}^{sj} \times \mathcal{M}_j (\ddot{\mathbf{R}}^j - \ddot{\mathbf{R}}^c) \right]$$

$$= \sum_{j\in\mathcal{B}} \left[ \dot{\mathbf{H}}^j + \sum_{s\in\mathcal{B}} \mathbf{D}^{sj} \times \mathcal{M}_j \ddot{\mathbf{R}}^j \right] \tag{341}$$

If $\mathbf{H}$ is written in terms of vector basis $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$, so that

$$\mathbf{H} = \mathbf{b}_1^0 H_1 + \mathbf{b}_2^0 H_2 + \mathbf{b}_3^0 H_3 = \{\mathbf{b}_1^0 \mathbf{b}_2^0 \mathbf{b}_3^0\} \begin{Bmatrix} H_1 \\ H_2 \\ H_3 \end{Bmatrix} \triangleq \{\mathbf{b}^0\}^T H \tag{342}$$

then

$$\dot{\mathbf{H}} = \{\mathbf{b}^0\}^T \dot{H} + \boldsymbol{\omega}^0 \times \{\mathbf{b}^0\}^T \mathbf{H} = \{\mathbf{b}^0\}^T \{\dot{H} + \tilde{\omega} H\} \tag{343}$$

and identification of Eq. (339) with Eq. (333) requires only a proof of the equality of $H$ and $T_{.\omega}$. This proof requires the following calculation for system angular momentum:

$$\mathbf{H} \triangleq \int \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\, dm = \sum_{j \in \mathcal{B}} \int_{\mathscr{b}_j} (\boldsymbol{\rho}^j + \mathbf{p}) \times (\dot{\boldsymbol{\rho}}^j + \dot{\mathbf{p}})\, dm$$

$$= \sum_{j \in \mathcal{B}} \left[ \mathcal{M}_j \boldsymbol{\rho}^j \times \dot{\boldsymbol{\rho}}^j + \int_{\mathscr{b}_j} \mathbf{p} \times \dot{\mathbf{p}}\, dm \right]$$

since $\int_{\mathscr{b}_j} \mathbf{p}\, dm = 0$ by mass center definition. Thus (with Eq. (310))

$$\mathbf{H} = \sum_{j \in \mathcal{B}} \left[ \mathcal{M}_j \boldsymbol{\rho}_j \times \dot{\boldsymbol{\rho}}_j + \int_{\mathscr{b}_j} \mathbf{p} \times (\boldsymbol{\omega}^j \times \mathbf{p})\, dm \right]$$

$$= \sum_{j \in \mathcal{B}} \left[ \mathcal{M}_j \left( \sum_{s \in \mathcal{B}} \mathbf{D}^{sj} \right) \times \left( \sum_{r \in \mathcal{B}} \dot{\mathbf{D}}^{rj} \right) + \mathbf{I}^j \cdot \boldsymbol{\omega}^j \right]$$

$$= \sum_{j \in \mathcal{B}} \left[ \mathcal{M}_j \sum_{s \in \mathcal{B}} \mathbf{D}^{sj} \times \left( \sum_{r \in \mathcal{B}} \boldsymbol{\omega}^r \times \mathbf{D}^{rj} \right) + \mathbf{I}^j \cdot \boldsymbol{\omega}^j \right] \tag{344}$$

The matrix $H$ representing $\mathbf{H}$ in vector basis $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$ therefore is given by

$$H = \sum_{j \in \mathcal{B}} \left[ \mathcal{M}_j \sum_{s \in \mathcal{B}} (C^{0s} D^{sj})^{\sim} \sum_{r \in \mathcal{B}} (C^{0r} \omega^r)^{\sim} C^{0r} D^{rj} + C^{0j} I^j C^{j0} C^{0j} \omega^j \right] \tag{345}$$

in which all vectors and dyadics have been represented by matrices in their local vector basis, and where

$$\{\mathbf{b}^s\} = C^{s0} \{\mathbf{b}^0\} \tag{346}$$

defines the direction cosine matrix relating a "local" set of dextral, orthogonal unit vectors $\mathbf{b}_1^s, \mathbf{b}_2^s, \mathbf{b}_3^s$ fixed in $\mathscr{b}_s$ to $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$ fixed in $\mathscr{b}_0$.

To show that Eq. (345) is also equal to $\overline{T}_{.\omega}$, we require

$$2\overline{T} \triangleq \int \dot{\boldsymbol{\rho}} \cdot \dot{\boldsymbol{\rho}}\, dm + \mathcal{M} \dot{\mathbf{R}}^c \cdot \dot{\mathbf{R}}^c = \sum_{j \in \mathcal{B}} \int_{\mathscr{b}_j} (\dot{\boldsymbol{\rho}}^j + \dot{\mathbf{p}}) \cdot (\dot{\boldsymbol{\rho}}^j + \dot{\mathbf{p}})\, dm + \mathcal{M} \dot{\mathbf{R}}^c \cdot \dot{\mathbf{R}}^c$$

$$= \sum_{j \in \mathcal{B}} \left[ \mathcal{M}_j \dot{\boldsymbol{\rho}}^j \cdot \dot{\boldsymbol{\rho}}^j + \int_{\mathscr{b}_j} \dot{\mathbf{p}} \cdot \dot{\mathbf{p}}\, dm \right] + \mathcal{M} \dot{\mathbf{R}}^c \cdot \dot{\mathbf{R}}^c$$

$$= \sum_{j\in\mathcal{B}} \left[ \mathcal{M}_j \dot{\boldsymbol{\rho}}^j \cdot \dot{\boldsymbol{\rho}}^j + \boldsymbol{\omega}^j \cdot \int_{\mathcal{A}_j} \mathbf{p} \times (\boldsymbol{\omega}^j \times \mathbf{p})\, dm \right] + \mathcal{M} \dot{\mathbf{R}}^c \cdot \dot{\mathbf{R}}^c$$

or

$$2\overline{T} = \sum_{j\in\mathcal{B}} [\mathcal{M}_j \dot{\boldsymbol{\rho}}^j \cdot \dot{\boldsymbol{\rho}}^j + \boldsymbol{\omega}^j \cdot \mathbf{I}^j \cdot \boldsymbol{\omega}^j] + \mathcal{M} \dot{\mathbf{R}}^c \cdot \dot{\mathbf{R}}^c \tag{347}$$

Substituting (from the derivative of Eq. (310))

$$\dot{\boldsymbol{\rho}}^j = \sum_{s\in\mathcal{B}} \boldsymbol{\omega}^s \times \mathbf{D}^{sj} \tag{348}$$

and (from Eq. (315))

$$\boldsymbol{\omega}^s = \boldsymbol{\omega}^0 + \sum_{r\in\mathcal{P}} \varepsilon_{rs} \dot{\gamma}_r \mathbf{g}^r \tag{349}$$

into $2\overline{T}$ yields

$$\begin{aligned} 2\overline{T} = \sum_{j\in\mathcal{B}} \mathcal{M}_j \Big[ \sum_{s\in\mathcal{B}} (\boldsymbol{\omega}^0 + \sum_{r\in\mathcal{P}} \varepsilon_{rs} \dot{\gamma}_r \mathbf{g}^r) \times \mathbf{D}^{sj} \Big] \cdot \Big[ \sum_{p\in\mathcal{B}} (\boldsymbol{\omega}^0 + \sum_{q\in\mathcal{P}} \varepsilon_{qp} \dot{\gamma}_q \mathbf{g}^q) \times \mathbf{D}^{pj} \Big] \\ + \sum_{j\in\mathcal{B}} (\boldsymbol{\omega}^0 + \sum_{r\in\mathcal{P}} \varepsilon_{rj} \dot{\gamma}_r \mathbf{g}^r) \cdot \mathbf{I}^j \cdot (\boldsymbol{\omega}^0 + \sum_{s\in\mathcal{P}} \varepsilon_{sj} \dot{\gamma}_s \mathbf{g}^s) + \mathcal{M} \dot{\mathbf{R}}^c \cdot \dot{\mathbf{R}}^c \end{aligned} \tag{350}$$

If all vectors and dyadics are written in terms of their local vector bases, and the direction cosine matrices defined by Eq. (346) are introduced, then Eq. (350) becomes

$$\begin{aligned} 2\overline{T} = \sum_{j\in\mathcal{B}} \mathcal{M}_j \Big[ \sum_{s\in\mathcal{B}} (\tilde{\omega}^0 + \sum_{r\in\mathcal{P}} \varepsilon_{rs} \dot{\gamma}_r C^{0r} \tilde{g}^r)\, C^{0s} D^{sj} \Big]^T \Big[ \sum_{p\in\mathcal{B}} (\tilde{\omega}^0 + \sum_{q\in\mathcal{P}} \varepsilon_{pq} \dot{\gamma}_q C^{0q} \tilde{g}^q)\, C^{0p} D^{pj} \Big] \\ + \sum_{j\in\mathcal{B}} (\omega^0 + \sum_{r\in\mathcal{P}} \varepsilon_{rj} \dot{\gamma}_r C^{0r} g^r)^T C^{0j} I^j C^{j0} (\omega^0 + \sum_{s\in\mathcal{B}} \varepsilon_{sj} \dot{\gamma}_s C^{0s} g^s) + \mathcal{M} \dot{\mathbf{R}}^c \cdot \dot{\mathbf{R}}^c \end{aligned} \tag{351}$$

By virtue of the anticommutativity of cross multiplication, and the skew symmetry of a matrix containing the tilde operator, the first matrix in square brackets may be written as

$$\begin{aligned} \Big[ \sum_{s\in\mathcal{B}} (\tilde{\omega}^0 + \sum_{r\in\mathcal{P}} \varepsilon_{rs} \dot{\gamma}_r C^{0r} \tilde{g}^r)\, C^{0s} D^{sj} \Big]^T &= \Big[ -\sum_{s\in\mathcal{B}} (C^{0s} D^{sj})^{\sim} (\omega^0 + \sum_{r\in\mathcal{P}} \varepsilon_{rs} \dot{\gamma}_r C^{0r} g^r) \Big]^T \\ &= \Big[ (\omega^{0T} + \sum_{r\in\mathcal{P}} \varepsilon_{rs} \dot{\gamma}_r g^{rT} C^{r0})\, (C^{0s} D^{sj})^{\sim} \Big] \end{aligned}$$

and similar operations can be performed on the second. In view of the fact that $\overline{T}$ is a scalar, and hence equal to its transpose, one can construct $\overline{T}_{,\omega}$ by formally taking that contribution to $\partial\overline{T}/\partial\omega^0$ that is made by the $\omega^{0T}$ appearing in the two premultiplier matrices appearing with superscripts $T$ in Eq. (351), and then doubling the result to obtain the complete expression for $\partial\overline{T}/\partial\omega^0$, or $\overline{T}_{,\omega}$. The result is

$$
\begin{aligned}
\bar{T}_{,\omega} &= \sum_{j\in\mathcal{B}} \mathcal{M}_j \sum_{s\in\mathcal{B}} (C^{0s}D^{sj})^{\sim} \sum_{p\in\mathcal{B}} [(\tilde{\omega}^0 + \sum_{q\in\mathcal{P}} \varepsilon_{pq}\dot{\gamma}_q C^{0q}\tilde{g}^q)\, C^{0p}D^{pj}] \\
&\quad + \sum_{j\in\mathcal{B}} C^{0j}I^jC^{j0}\,(\omega^0 + \sum_{s\in\mathcal{P}} \varepsilon_{sj}\dot{\gamma}_s C^{0s}g^s) \\
&= \sum_{j\in\mathcal{B}} \mathcal{M}_j\,(C^{0s}D^{sj})^{\sim} \sum_{p\in\mathcal{B}} (C^{0p}\omega^p)^{\sim} C^{0p}D^{pj} \\
&\quad + \sum_{j\in\mathcal{B}} C^{0j}I^jC^{j0}C^{0j}\omega^j \equiv H
\end{aligned}
\tag{352}
$$

as indicated by Eq. (345). Thus we have established that Eqs. (339), (333), (323a), and (308a) are all identical.

*d. Summary for the point-connected bodies in a tree.* It has been proven that, if we freely use the Hooker–Margulies kinematical identity found in Eq. (310), then exactly the same equations of motion for the point-connected set of rigid bodies in a topological tree emerge from each of the following derivation procedures:

(1) The Hooker–Margulies/Hooker equations.

(2) Kane's quasi-coordinate formulation of D'Alembert's principle.

(3) The combination of Lagrange's generalized coordinate equations and Lagrange's quasi-coordinate equations.

(4) The combination of Lagrange's generalized coordinate equations and the vector rotational equation $\mathbf{M} = \dot{\mathbf{H}}$ applied to the total system and resolved into a vector basis fixed in the reference body.

Any comparisons to be drawn among these methods must be based on subjective judgments of relative ease of formulation, since the digital computer numerical integration task is exactly the same in every case. The application of some method other than the four considered here might conceivably produce different equations permitting more efficient integration. Russell's derivation algorithm (Ref. 18) seems to provide the alternative in the present literature most worthy of further consideration; the first order form of Lagrange's equations (Eqs. (155) and (156)) advocated in Ref. 35 seems computationally preferable to the second order Lagrangian form, but this is a low standard of comparison.

## B. Rigid-Elastic Body System Models

**1. Single elastic body with small deformations.** As stated in Subsection II-A-2, it is both commonplace and reasonable to develop approximate descriptions of the motion of an elastic continuum in terms of a finite number of degrees of freedom, and thereby to represent that motion in terms of ordinary rather than partial differential equations. Such approximations are rarely challenged by those with engineering responsibilities, because in many applications this is the only avenue to meaningful results. Much more controversial is the procedure by which the limited number of degrees of freedom are selected. Very frequently in the literature this problem is sidestepped (as it has been thus far in this report) by simply postulating that somehow someone has provided the modal vectors (or "mode shapes") that correspond to the deformation coordinates, so that one can (as in Eq. (31)) simply expand the vector $\mathbf{u}(r_1, r_2, r_3, t)$ that represents the displacement of a material point $(r_1, r_2, r_3)$ relative to some assigned position fixed in some reference frame $f$ as

$$\mathbf{u}(r_1, r_2, r_3, t) = \sum_{j=1}^{\bar{n}} \boldsymbol{\varphi}^j (r_1, r_2, r_3) \, q_j(t) \tag{353}$$

with $\boldsymbol{\varphi}^j (r_1, r_2, r_3)$ given for $j = 1, \cdots, \bar{n}$.

The selection of an equation formulation procedure depends crucially on the question of whether or not one treats the modal vectors as given. If one imagines that these vectors are somehow provided as part of the problem statement, then equations of motion can be constructed directly in terms of the generalized coordinates of deformation $q_1, \cdots, q_{\bar{n}}$ and the six scalars that define the translation and rotation of the reference frame $f$ with respect to which deformations are measured (see Subsection II-A-2). Lagrange's equations or Hamilton's equations then could be used to obtain equations of motion in these generalized coordinates directly, without ever considering the primitive deformations represented by the vector $\mathbf{u}$. This task would be facilitated by the assumption of small deformations, which permit terms above the *second degree* in the kinetic and potential energy expressions to be abandoned.

If the modal vectors are given, it is still possible to formulate equations of motion from Newton's second law or Lagrange's form of D'Alembert's principle, but with these methods one would be obliged to state the equations of motion first in terms of the primitive deformation vector $\mathbf{u}(\mathbf{r}, t)$, and then to introduce Eq. (353) as a coordinate transformation to obtain equations of motion involving the generalized coordinates $q_1, \cdots, q_{\bar{n}}$.

As we have established in general terms, the results of the application of Lagrange's generalized coordinate equations must be the same as those obtained from Lagrange's form of D'Alembert's principle. For a single elastic body subject to small deformations and arbitrary gross motions, the equations of motion that emerge from either of these Lagrangian formulations are given by Eq. (41). Moreover, if all of the generalized coordinates represent small deviations from a solution to the equations of motion, then the linearized equations are given by Eq. (117). Thus *if the modal vectors are assumed known*, then it seems to be advantageous to adopt one of these two Lagrangian approaches, and to accept Eq. (41) or Eq. (117) as the equation of motion.

On the other hand, if the analyst has the responsibility of calculating the modal vectors as part of his equation formulation task, then the advantages of Lagrange's equation largely disappear. The modal vectors must then themselves be obtained from a set of equations of motion, which must be written in terms of $\mathbf{u}(\mathbf{r}, t)$ directly. These equations can be obtained in either of two quite different forms (see Ref. 41).

If the elastic body has a very simple configuration and simple patterns of gross motion, then it might be idealized in classical terms as a beam, plate, or shell and its equations of motion might be written as linear partial differential equations, obtained either from Hamilton's principle or from the application of Newton's second law to differential elements of mass. These equations might then be separable, in the sense defined in Subsection II-A-2 and represented by Eq. (353), and the separated linear equations might then be solved (literally or numerically) for the modal vectors, which would then appear as selections from among the infinity of eigenvectors of the system equations.

In the vast majority of cases of practical interest, the partial differential equations of motion of an elastic body would be very difficult to formulate and even more difficult to solve. In such cases it is necessary for the analyst to "discretize" the mathematical model of the elastic body, introducing a model that has a finite number of degrees of freedom even before he formulates any equations of motion. In the discretized model all mass might be concentrated in the form of particles located at a finite number of *nodes* of the elastic body, or there may be little rigid bodies at the nodes, as in Ref. 19. Mass might even be distributed throughout the body in the form of distributed-mass internodal elements (*finite elements*, as in Ref. 27), but the kinematical variables would be limited to those associated with the nodes, and the system would still have a finite number of degrees of freedom. In every case the equations of motion would have to be linear, constant coefficient, ordinary differential equations to permit the calculation of the modal vectors as the eigenvectors of the system equations. The process of modal coordinate transformation has been treated extensively in earlier reports in this series (see Refs. 19, 27, and 31), and will not be reviewed here. Our present concern is solely with the comparison of procedures for formulating equations of motion.

Once the commitment has been made to derive equations of motion of a discretized model of an elastic body to obtain the modal vectors required for a modal coordinate transformation, then it would appear that the most efficient approach involves the direct application of $\mathbf{F} = m\mathbf{A}$ and $\mathbf{M} = \dot{\mathbf{H}}$ to nodal bodies and to the total system. This approach is more efficient than Lagrange's generalized coordinate equations because in the latter one must in general retain in the kinetic energy all second degree terms in small variables, and this can significantly complicate the problem of kinematic analysis. With a direct Newton–Euler approach one can linearize in small variables whenever they appear.

It is not so obvious that a direct Newton–Euler approach is preferable to the application of D'Alembert's principle in the form advanced by Lagrange or by Kane, or to the Lagrangian quasi-coordinate approach. This range of possibilities is dealt with completely by the following propositions, applicable to a discretized model of an elastic body consisting of $n$ rigid nodal bodies interconnected by a massless elastic structure subject to small deformations.

*Proposition 1.* Applying $\mathbf{F} = m\mathbf{A}$ and $\mathbf{M} = \dot{\mathbf{H}}$ to nodal bodies and recording scalar equations in a vector basis fixed in some reference frame $f$ following the gross motion, with deformation variables consisting of translations and rotations of nodal bodies relative to $f$, produces the same equations of motion that emerge from Lagrange's equations and from Lagrange's form of D'Alembert's principle when the generalized coordinates are the translations and rotations of nodal bodies relative to $f$, referred to axes fixed in $f$.

*Proposition 2.* Applying $\mathbf{M} = \dot{\mathbf{H}}$ to a discretized total elastic body and recording scalar equations for a vector basis fixed in some gross motion frame $f$ in which the system mass center is fixed, including the inertial angular velocity $\boldsymbol{\omega}$ of $f$ among the variables, produces the same three equations of motion that emerge from either Kane's or Lagrange's quasi-coordinate formulation, with the scalar components of $\boldsymbol{\omega}$ for a vector basis fixed in $f$ chosen as the quasi-coordinate derivatives.

A satisfactory proof of Proposition 1 can be presented without lengthy derivations and comparisons. We have already established in completely general terms that Lagrange's equations for independent generalized coordinates (see Eqs. (70)) are identical to those obtained from Lagrange's form of D'Alembert's principle as applied to systems with independent generalized coordinates (see Eq. (67) or Eq. (11), or Eq. (28)). From Eq. (28) it is clear that if the generalized coordinates $q_k$ are chosen for $k = 1, \cdots, 6n$ to represent the $3n$ translations $u_i^j$ of the mass centers of the $n$ nodal bodies along axes fixed in $f$ and the $3n$ orthogonal rotations $\beta_i^j$ of these bodies ($j = 1, \cdots, n$ and $i = 1, 2, 3$), then the quantities $\partial\dot{\mathbf{R}}_j/\partial\dot{q}_k$ and $\partial\boldsymbol{\omega}^j/\partial\dot{q}_k$ must correspond to orthogonal unit vectors $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ fixed in $f$, and Eq. (28) produces exactly the same equations as would emerge from

$$\mathbf{f}_i \cdot \mathbf{F}^j = \mathcal{M}_j \ddot{\mathbf{R}}^j \cdot \mathbf{f}_j, \qquad j = 1, \cdots, n \text{ and } i = 1, 2, 3$$

$$\mathbf{f}_i \cdot \mathbf{M}^j = \dot{\mathbf{H}}^j \cdot \mathbf{f}_i, \qquad j = 1, \cdots, n \text{ and } i = 1, 2, 3$$

Thus Proposition 1 is proven.

The proof of Proposition 2 requires more development. The objective is to demonstrate that

$$\mathbf{f}_i \cdot (\mathbf{M} - \dot{\mathbf{H}}) = 0 \tag{354}$$

and Lagrange's quasi-coordinate equations in the form of Eq. (333) are identical, with the 3 by 1 matrix $\omega$ representing the inertial angular velocity of $f$ in the vector basis fixed in $f$, and the 3 by 1 matrix $q^0$ containing a set of inertial attitude angles for $f$. The kinetic energy $\overline{T}$ now is a function of $\omega$ and of the various scalar measures of deformation relative to $f$, as represented generically by $q_1, \cdots, q_{6n}$ or explicitly by $u_i^j$ and $\beta_i^j$ for $j = 1, \cdots, n$ and $i = 1, 2, 3$. The matrix $W_0$ has the meaning established by Eq. (334), and the 3 by 1 matrix of generalized forces $Q^0$ has the elements (from Eq. (22))

$$Q_k^0 = \sum_{j=1}^{n} \left( \frac{\partial \dot{\mathbf{R}}^j}{\partial \dot{q}_k^0} \cdot \mathbf{f}^j + \frac{\partial \boldsymbol{\omega}^j}{\partial \dot{q}_k^0} \cdot \mathbf{m}^j \right) \tag{355}$$

The identity of Eqs. (333) and (354) will be proven by first showing that

$$W_0^{-1} Q^0 = \{\mathbf{f}\} \cdot \mathbf{M} \tag{356}$$

and then showing that

$$\frac{d}{dt}(\overline{T}_{,\omega}) + \tilde{\omega}\overline{T}_{,\omega} - W_0^{-1}\overline{T}_{,q^0} = \{\mathbf{f}\} \cdot \dot{\mathbf{H}} \tag{357}$$

where

$$\{\mathbf{f}\} \triangleq \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{Bmatrix}$$

In Eq. (355) we can substitute

$$\boldsymbol{\omega}^j = \boldsymbol{\omega} + \{\mathbf{f}\}^T \dot{\beta}^j$$

and

$$\dot{\mathbf{R}}^j = \dot{\mathbf{R}}^c + \boldsymbol{\omega} \times \bar{\boldsymbol{\rho}}^j + \{\mathbf{f}\}^T \dot{u}^j$$

where $\boldsymbol{\omega}$ is the inertial angular velocity of $f$, and other symbols are as previously defined. Since in these expressions only $\boldsymbol{\omega}$ depends upon $q_k^0$ $(k = 1, 2, 3)$, Eq. (355) can be written as

$$\begin{aligned} Q_k^0 &= \sum_{j=1}^{n} \left( \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k^0} \times \boldsymbol{\rho}^j \cdot \mathbf{f}^j + \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k^0} \cdot \mathbf{m}^j \right) = \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k^0} \cdot \sum_{j=1}^{n} (\boldsymbol{\rho}^j \times \mathbf{f}^j + \mathbf{m}^j) \\ &= \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k^0} \cdot \mathbf{M} \end{aligned} \tag{358}$$

where $\mathbf{M}$ is the moment of external forces about the system mass center.

In matrix terms in vector basis $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$, Eq. (358) becomes

$$Q^0 = \omega_{,\dot{q}^0}^T M$$

where $M = \{\mathbf{f}\} \cdot \mathbf{M}$ so that with Eq. (334) the validity of Eq. (356) is established.

The proof of Eq. (357) requires firstly the recognition that $\bar{T}_{;q^0} = 0$, since the inertial attitude angles of $f$ cannot appear in $\bar{T}$, and secondly the realization that with $\mathbf{H} = \{\mathbf{f}\}^T H$ one can write the right hand side as $\dot{H} + \tilde{\omega} H$. Thus Eq. (357) can be proven if the relationship

$$H = \bar{T}_{,\omega} \tag{359}$$

can be established.

The left side of Eq. (359) is available from Eq. (344) in the form

$$H = \{\mathbf{f}\} \cdot \mathbf{H} = \{\mathbf{f}\} \cdot \sum_{j=1}^{n} [\mathcal{M}_j \boldsymbol{\rho}_j \times \dot{\boldsymbol{\rho}}_j + \mathbf{I}^j \cdot \boldsymbol{\omega}^j] \tag{360}$$

Substituting

$$\dot{\boldsymbol{\rho}}_j = \boldsymbol{\omega} \times \boldsymbol{\rho}_j + \{\mathbf{f}\}^T \dot{u}^j = \{\mathbf{f}\}^T (\tilde{\omega} \rho_j + \dot{u}^j) \tag{361}$$

$$\boldsymbol{\omega}_j = \boldsymbol{\omega} + \{\mathbf{f}\}^T \dot{\beta}^j = \{\mathbf{f}\}^T (\omega + \dot{\beta}^j) \tag{362}$$

and

$$\mathbf{I}^j = \{\mathbf{b}^j\}^T I^j \{\mathbf{b}^j\} = \{\mathbf{f}\}^T C^{fj} I^j C^{jf} \{\mathbf{f}\} \tag{363}$$

where $\{\mathbf{b}^j\}$ is a vector basis fixed in the $j$th nodal body and

$$\{\mathbf{b}^j\} = C^{jf} \{\mathbf{f}\} \qquad \text{and} \qquad C^{fj} = [C^{jf}]^T \tag{364}$$

one finds

$$H = \sum_{j=1}^{n} [\mathcal{M}_j \tilde{\rho}_j (\tilde{\omega} \rho_j + \dot{u}^j) + C^{fj} I^j C^{jf} (\omega + \dot{\beta}^j)] \tag{365}$$

The right side of Eq. (359) requires an expression for the kinetic energy of a system of rigid bodies, which from Eq. (74) is

$$\bar{T} = \frac{1}{2}\sum_{j=1}^{n}[\mathcal{M}_j\dot{\mathbf{R}}^j\cdot\dot{\mathbf{R}}^j + \boldsymbol{\omega}^j\cdot\mathbf{I}^j\cdot\boldsymbol{\omega}^j]$$

$$= \frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^c\cdot\dot{\mathbf{R}}^c + \frac{1}{2}\sum_{j=1}^{n}[\mathcal{M}_j\dot{\boldsymbol{\rho}}_j\cdot\dot{\boldsymbol{\rho}}_j + \boldsymbol{\omega}^j\cdot\mathbf{I}^j\cdot\boldsymbol{\omega}^j] \tag{366}$$

Substitution of Eqs. (361) through (363) provides

$$\bar{T} = \frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^c\cdot\dot{\mathbf{R}}^c + \frac{1}{2}\sum_{j=1}^{n}[\mathcal{M}_j(\tilde{\omega}\rho_j + \dot{u}^j)^T(\tilde{\omega}\rho_j + \dot{u}^j)$$

$$+ (\omega + \dot{\beta}^j)^T C^{fj}I^jC^{jf}(\omega + \dot{\beta}^j)]$$

$$= \frac{1}{2}\mathcal{M}\dot{\mathbf{R}}^c\cdot\dot{\mathbf{R}}^c + \frac{1}{2}\sum_{j=1}^{n}[\mathcal{M}_j(\omega^T\tilde{\rho}_j + \dot{u}^{jT})(\tilde{\omega}\rho_j + \dot{u}^j)$$

$$+ (\omega^T + \dot{\beta}^{jT})C^{fj}I^jC^{jf}(\omega + \dot{\beta}^j)] \tag{367}$$

In view of the fact that $\bar{T}$ is a scalar, and hence equal to its transpose, one can construct $\bar{T}_{,\omega}$ by formally taking that contribution to $\bar{T}_{,\omega}$ that is made by the $\omega^T$ appearing in the two premultiplier matrices and doubling the result, to find

$$\bar{T}_{,\omega} = \sum_{j=1}^{n}[\mathcal{M}_j\tilde{\rho}_j(\tilde{\omega}\rho_j + \dot{u}^j) + C^{fj}I^jC^{jf}(\omega + \dot{\beta}^j)] \tag{368}$$

Since by Eqs. (368) and (355) Eq. (359) is proven, then Eq. (357) is also established, and with Eq. (356) Proposition 2 is proven.

With the help of Propositions 1 and 2 we can see that for a discretized model of an elastic body most of the primary procedures for equation formulation yield results that are not only equivalent, but identical. Continued debate about the relative merits of various derivation procedures then becomes pointless.

2. **Interconnected rigid bodies and elastic bodies.** The propositions of the previous section can be generalized to provide parallel propositions that apply to any holonomic system of rigid bodies and elastic bodies. In parallel with Proposition 2 of the section on completely elastic bodies we have the following.

*Proposition* 3. Applying $\mathbf{M} = \dot{\mathbf{H}}$ to any material continuum and recording scalar equations for a vector basis fixed in any reference frame $f$ in which the system mass center is fixed, including the inertial angular velocity $\boldsymbol{\omega}$ of $f$ among the variables, produces the same three equations of motion that emerge from either Kane's or Lagrange's quasi-coordinate formulation, with the scalar components of $\boldsymbol{\omega}$ for a vector basis fixed in $f$ chosen as the quasi-coordinate derivatives.

The proof of Proposition 3 follows that of Proposition 2, consisting again of a demonstration of the validity of Eqs. (356) and (357). The fundamental difference lies in the use of integral expressions for $Q_k^0$, $\mathbf{M}$, $\mathbf{H}$, and $\bar{T}$.

Eq. (355) becomes, from Eq. (32a) and (17),

$$Q_k^o = \int \frac{\partial \dot{\mathbf{R}}}{\partial \dot{q}_k^o} \cdot d\mathbf{f} = \int \frac{\partial}{\partial \dot{q}_k^o} (\dot{\mathbf{R}}^c + \boldsymbol{\omega} \times \bar{\boldsymbol{\rho}} + \dot{\mathbf{u}}) \cdot d\mathbf{f}$$
$$= \int \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k^o} \times \boldsymbol{\rho} \cdot d\mathbf{f} = \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k^o} \int \boldsymbol{\rho} \times d\mathbf{f} = \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_k^o} \cdot \mathbf{M} \tag{369}$$

which corresponds to Eq. (358), and confirms Eq. (356).

Proof of Eq. (357) now again reduces to proof of Eq. (359), where now from Eq. (43)

$$\mathbf{H} = \mathbf{I} \cdot \boldsymbol{\omega} + \int (\bar{\boldsymbol{\rho}} + \mathbf{u}) \times \mathring{\mathbf{u}}\, dm$$
$$= \{\mathbf{f}\}^T \left[ I\omega + \int (\tilde{\bar{\rho}} + \tilde{u})\, \dot{u}\, dm \right] = \{\mathbf{f}\}^T H \tag{370}$$

and from Eq. (76) with $p$ replaced by $c$,

$$\bar{T} = \frac{1}{2} \mathcal{M} \dot{\mathbf{R}}^c \cdot \dot{\mathbf{R}}^c + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \int (\bar{\boldsymbol{\rho}} + \mathbf{u}) \times \mathring{\mathbf{u}}\, dm + \frac{1}{2} \int \mathring{\mathbf{u}} \cdot \mathring{\mathbf{u}}\, dm$$
$$= \frac{1}{2} \mathcal{M} \dot{\mathbf{R}}^c \cdot \dot{\mathbf{R}}^c + \frac{1}{2} \omega^T I \omega + \omega^T \int (\tilde{\bar{\rho}} + \tilde{u})\, \dot{u}\, dm + \frac{1}{2} \int \dot{u}^T \dot{u}\, dm$$

Thus

$$\bar{T}_{,\omega} = I\omega + \int (\tilde{\bar{\rho}} + \tilde{u})\, \dot{u}\, dm \tag{371}$$

and comparison with Eq. (370) confirms Eq. (359), and hence establishes Eq. (357) and proves Proposition 3.

Proposition 3 is concerned with only three of the equations of motion, namely those that describe the rotation of the reference frame $f$, which defines the gross motion of the nonrigid system. Equations of translation of the system mass center, which is fixed in $f$, are almost universally obtained by simply applying Newton's second law to the entire system. Any remaining controversy is thus limited to discussions of the relative advantages of various procedures for obtaining equations of motion of internal components relative to frame $f$. What follows is intended to contribute to the resolution of this final controversy.

*Proposition 4.* For any dynamical system modeled as a finite collection of particles and rigid bodies, described by independent coordinates that define either (1) the rotation of a rigid body relative to a point-connected rigid body, or (2) the translation of a particle relative to a rigid body with respect to which its motion is constrained, or (3) the translation or rotation of a particle or rigid body with no kinematical constraints relative to some reference frame $f$, in terms of an orthogonal vector basis fixed in $f$, the equations of motion in these variables may be obtained from Lagrange's form of D'Alembert's principle, as in Eq. (28). Moreover, identical equations will result from the application of Lagrange's equations, as in Eq. (70), or from the application of Kane's method, as in Eq. (66). Finally, any equations of motion that are obtained by applying $\mathbf{F} = m\mathbf{A}$ and $\mathbf{M} = \dot{\mathbf{H}}$ to individual nodal bodies free of kinematical constraints will be identical to a subset of those obtained from Eq. (28), assuming the same selection of coordinates describing translations and rotations of the nodal bodies relative to frame $f$.

Proof of Proposition 4 requires no more than the integration of arguments already advanced in this report. Lagrange's form of D'Alembert's principle has been presented in this report in many guises, including Eq. (67) and (for systems of particles and rigid bodies) Eq. (28). Lagrange's equations for independent generalized coordinates (Eq. 70) were derived from Eq. (67), and have been shown to provide identical equations for any particular application. Thus Eq. (28) and Eq. (70) provide identical results. Moreover, it has already been established that Kane's quasi-coordinate equations (Eq. (66)) are identical to Lagrange's form of D'Alembert's principle when the quasi-coordinates are generalized coordinates. Proof of Proposition 4 thus requires only the observation that for an unconstrained nodal body labeled $r$ the equations

$$\mathbf{f}_i \cdot (\mathbf{F}^r - \mathcal{M}_r \ddot{\mathbf{R}}^r) = 0 \tag{372}$$

and

$$\mathbf{f}_i \cdot (\mathbf{M}^r - \dot{\mathbf{H}}^r) = 0 \tag{373}$$

for $i = 1, 2, 3$ will also emerge from Eq. (28). This result is obvious in the light of Eqs. (11) through (13) if for some $k$, $r$, and all $s \neq r$,

$$\frac{\partial \boldsymbol{\omega}^r}{\partial \dot{q}_k} = 0 \qquad \frac{\partial \dot{\mathbf{R}}^r}{\partial \dot{q}_k} = \mathbf{f}_i \qquad (i = 1, 2, 3)$$

$$\frac{\partial \boldsymbol{\omega}^s}{\partial \dot{q}_k} = 0 \qquad \frac{\partial \dot{\mathbf{R}}^s}{\partial \dot{q}_k} = 0$$

and this will clearly be the case if among the variables $q_1, \cdots, q_n$ there appear scalars that describe the translations of the nodal body relative to $f$ in vector basis $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ and the rotations of the nodal body relative to $f$ about axes parallel to $\mathbf{f}_1, \mathbf{f}_2$, and $\mathbf{f}_3$. Thus Proposition 4 is proven.

Proposition 3 applies to any material continuum, but Proposition 4 is more restricted, being limited to combinations of rigid bodies and discretized elastic bodies. We should also establish and compare the alternative equation formulation procedures applicable to an arbitrary continuum that is described in terms of a set of distributed or modal coordinates, for which mode shapes are known (see Eq. 353). In this case we have already established in Eq. (42) a complete set of equations of motion from Lagrange's form of D'Alembert's principle. Since we know that Kane's approach and Lagrange's generalized coordinate equations must yield results identical to those that follow from Eq. (42), we have no realistic alternative, and need not look beyond these equations.

## IV. Conclusions and Recommendations

As noted in the Introduction, it is the purpose of this study to explore three of the basic methods of analytical dynamics and their variations, and to examine them for their suitability for the development of multipurpose generic formulations of the equations of motion of nonrigid spacecraft. This report must conclude with an assessment of the advisability of making the commitment of resources necessary to develop the generic computer programs required for spacecraft simulation based on the equations of analytical dynamics examined here. This assessment depends not only on what possibilities are offered by the methods of analytical dynamics but also on the alternatives, which have not been given detailed exposition in this report, but which have been covered extensively in previous JPL reports by this writer. Thus in what follows there appear brief comparisons of generic computer programs now operational or in development at JPL with generic computer programs that could be developed using the methods of analytical dynamics.

The point-connected set of rigid bodies in a topological tree provides the most straightforward class of spacecraft idealizations for which generic computer programs have been written. Reference 46 documents the computer subroutines most recently developed at JPL, and soon to be included in the COSMIC library, which is available to all qualified users. This program is based on Ref. 31, which in turn is an outgrowth of the work of Hooker (Ref. 26) and of Hooker and Margulies (Ref. 13). The equations are therefore referred to here as the HMH equations. The existing and generally available program based on the HMH equations presents the standard against which any proposed new program should be measured.

As shown in Subsection III-A-4, the HMH equations in the standard program are precisely the same as would emerge from the two quasi-coordinate methods identified here with Lagrange and Kane, and these are identical to those that would be obtained by combining Lagrange's equations for independent generalized coordinates with the vector rotational equation $\mathbf{M} = \dot{\mathbf{H}}$, assuming that the same coordinates are used throughout. The alternative of employing Hamilton's equations seems less acceptable, because of the necessity of taking partial derivatives of a matrix inverse. Either the matrix must be inverted and differentiated by hand, or the digital computer must perform this task; in the latter case one must either employ symbolic manipulation or resort to the substitution of matrix identities that add to the computational burden.

Thus we can answer negatively the central question, "should we develop new computer programs to replace the HMH equations with alternatives considered here?" Two reservations should however be noted explicitly. First it should be noted that the negative conclusion advanced here does not preclude the possibility that a completely different method might be developed that supersedes the HMH equations. Russell's equations (Ref. 18) have been mentioned as a candidate for this role, although these equations have not been cast in generic form and fully automated for simulation as have the HMH equations. The second reservation relates to the fact that we have not addressed the detailed problems of programming a given set of equations for numerical integration. We have not entered the controversy over the desirability of transforming second order equations into first order form before integration, nor considered how this should be done if it is desirable. The work of Vance and Sitchin (Ref. 10) does deal with this problem; they recommend a first order form of Lagrange's equations represented here by Eqs. (153) and (154) rather than the second order form represented here by Eq. (110), and on this essentially computational issue no position is taken in this report. The contribution by Bodley and Park (Ref. 34) is in the same category; having obtained quasi-coordinate equations that would if developed be identical to the HMH equations, they transform to generalized momenta, presumably for real or imagined computational convenience. We have not examined here the computational advantages or disadvantages of this transformation.

The transition from the multiple-rigid-body tree model to an arbitrary collection of particles and rigid bodies takes us beyond the range of any competitive generic computer program, so we no longer have a standard against which to measure new programs, and judgments become more subjective. If this investigator were required to generalize the multiple-rigid-body tree model to some more general model involving only particles and rigid bodies, he would adopt Kane's quasi-coordinate approach (Refs. 36 and 40). This method provides the HMH equations for the multiple-rigid-body tree, with the help of the kinematical relationship in Eq. (310); it automatically produces equations of minimum dimension, even for simple nonholonomic systems; and it is straightforward and physically interpretable in application.

Generalization of the spacecraft mathematical model to include deformable bodies raises many new issues. As noted in Subsection III-B-1, one must decide in discussing the dynamics of deformable bodies whether or not the "mode shapes" are to be treated as given or considered to be among the unknowns of the problem. Although you can write a lovely paper based on the supposition that the mathematical model comes to you complete with a finite number of mode shapes, you cannot expect this luxury when your objective is to launch a successful spacecraft. Even if the mode shapes for the spacecraft are "given," the gift should be accepted with caution and with critical inquiry into its origins. It is therefore strongly recommended here that the task of selecting the mode shapes and the number of modal coordinates be considered to be an important phase (perhaps the most important phase) of any flexible spacecraft simulation effort. In most cases, this charge implies a return to a mathematical model that is either a very simple continuum (such as a beam) or a discretized system of nodal bodies and finite elements. In the case of the continuum one must obtain partial differential equations of motion (perhaps by means of Hamilton's principle, as in Ref. 41) and separate them into ordinary differential equations. For most spacecraft applications, however, a discretized model is required, and the methods of this report must be compared to alternatives presented in previous JPL Reports in this series (see Refs. 19, 27, 32).

Reference 32 provides a generic formulation of the equations of motion of a topological tree of point-connected rigid bodies having nonrigid appendages. For the special case of small-deformation elastic appendages with mass concentrated in the form of nodal bodies, these equations are being programmed at JPL at this writing. When completed, this program will become a part of the COSMIC library, and will be generally available as a standard against which other methods with similar restrictions can be prepared.

In trying to determine whether or not the methods of analytical dynamics examined in this report offer advantages over existing procedures based on a Newton–Euler formulation, we can rely heavily on Propositions 3 and 4 of Subsection III-B-2. Proposition 3 indicates in more precise terms that by applying $\mathbf{M} = \dot{\mathbf{H}}$ to an arbitrary material system we get exactly the same equations that would be obtained by applying either Kane's or Lagrange's quasi-coordinate equations with the same choice of variables. Proposition 4 indicates in essence that for a discretized model of an elastic body the application of $\mathbf{F} = m\mathbf{A}$ and $\mathbf{M} = \dot{\mathbf{H}}$ to nodal bodies and to point-connected rigid bodies as in Ref. 32 produces precisely the same equations as would be obtained by Lagrange's form of D'Alembert's principle, or by Lagrange's generalized coordinate equations and Lagrange's quasi-coordinate equations, or by Kane's equations. As in the multiple-rigid-body case, Hamilton's equations seem not to be competitive. Again there appears to be no demonstrable advantage in any of the methods of analytical dynamics over the Newton–Euler methods used in deriving equations for the standard program now under development.

Generalization beyond the standard program could probably be accomplished by any of several methods without influencing the final result. If the task were undertaken by this analyst, he would elect to work with Kane's formulation (Refs. 36 and 40), which in many applications reduces to the use of D'Alembert's principle or the Newton–Euler equations, but which offers advantages in systematic constraint elimination and variable reduction.

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# Appendix A

## Notational Conventions

In formulating equations of motion of dynamical systems of arbitrary dimension, it is most convenient to use matrices. The resulting equations have the added advantage of amenability to programming for numerical integration. Unfortunately, with these advantages comes the burden of remembering notational conventions, without which the analysis becomes too cumbersome and complex to be intelligible.

A rather complete symbol list appears under Definition of Symbols, Appendix B, and individual symbols are also defined as they are introduced. In addition, the basic conventions applied in this report to whole *classes* of symbols should be noted as follows:

(1) The symbol $\triangleq$ means *equals by definition*, and $\equiv$ means *identically equal to* or *equal as a consequence of definitions*. Of course $=$ merely indicates equality.

(2) Gibbsian vectors and dyadics are boldface letters (as in the vector $\mathbf{R}$ and the dyadic $\mathsf{I}$); letters representing dyadics are *sans serif* (as in $\mathsf{U}$ and $\mathsf{D}$).

(3) Scalars are either subscripted italic letters (as in $m_j$ and $q_j$) or script letters (as in $\mathcal{L}$, $\mathcal{J}$, and $\mathcal{H}$).

(4) Matrices are generally either unadorned (as in $q$ and $A$) or superscripted letters (as in $\omega^j$). Matrix transposition is indicated by superscript $T$. A row matrix is always written as the transpose of a column matrix. Thus the symbol $q$ *cannot* represent a row matrix; the row matrix with elements $q_1, \cdots, q_n$ must be written as $q^T$.

(5) Braces are used to identify a column or row matrix in terms of its elements (which may be scalars or matrix partitions separated by dashed lines). Thus we might find

$$q \triangleq \begin{Bmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{Bmatrix} \qquad \text{and} \qquad q^T = \{q_1, \cdots, q_n\}$$

and we might find the column matrix $x$ written as

$$x \triangleq \begin{Bmatrix} q \\ \cdots \\ p \end{Bmatrix}$$

where $q$ and $p$ are column matrices establishing partitions of $x$. Braces may also be used simply to emphasize the matrix structure implied by a symbol. The column matrix $q$ of elements $q_1, \cdots, q_n$ might be written as $\{q_j\}$ or even $\{q\}$ to provide a reminder that $q$ is a column matrix with elements $q_1, q_2 \cdots$.

Braces also enclose column arrays of Gibbsian vectors, which arrays are never written without braces.

(6) Square brackets are used to identify the elements (scalar or matrix partitions) of a matrix other than a column matrix or row matrix. For example

$$A \triangleq [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

and

$$P = \left[\begin{array}{c|c} P^{11} & P^{12} \\ \hline P^{21} & P^{22} \end{array}\right]$$

(7) Gibbsian vector and dyadic differentiation (ordinary or partial) is referred to an inertial reference frame unless otherwise stated; in particular, a dot over a Gibbsian vector or dyadic indicates ordinary time differentiation in an inertial reference frame. A dot over a scalar is an ordinary time derivative, and a dot over a matrix indicates time differentiation of its scalar elements.

(8) Partial differentiation of one scalar with respect to another is sometimes indicated by the comma convention, so that

$$\mathcal{L}_{,t} \triangleq \frac{\partial \mathcal{L}}{\partial t}$$

(9) Partial differentiation of a scalar with respect to a column matrix is the column matrix of partial derivatives with matching indices. For example

$$\frac{\partial \mathcal{L}}{\partial q} \triangleq \left\{\frac{\partial \mathcal{L}}{\partial q_j}\right\} \triangleq \begin{Bmatrix} \dfrac{\partial \mathcal{L}}{\partial q_1} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial \mathcal{L}}{\partial q_n} \end{Bmatrix}$$

The comma convention may be used for matrices, so that

$$\mathcal{L}_{,q} \triangleq \frac{\partial \mathcal{L}}{\partial q}$$

(10) Partial differentiation of a scalar with respect to a row matrix is the row matrix of partial derivatives with matching indices. For example

$$\frac{\partial \mathcal{L}}{\partial q^T} \triangleq \left\{\frac{\partial \mathcal{L}}{\partial q_j}\right\}^T \triangleq \left\{\frac{\partial \mathcal{L}}{\partial q_1} \cdots \frac{\partial \mathcal{L}}{\partial q_n}\right\} \triangleq \mathcal{L}_{,q^T}$$

Note that $\mathcal{L}_{,q^T} = \{\mathcal{L}_{,q}\}^T$.

(11) Partial differentiation of a 1 by $n$ row matrix $q^T$ by an $m$ by 1 column matrix $x$ is the $m$ by $n$ matrix of partial derivatives with element $\partial q_j^T/\partial x_i$ in the $i$th row, $j$th column. The comma convention may be used. Thus

$$q^T_{,x} \triangleq \frac{\partial q^T}{\partial x} \triangleq \begin{bmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_2}{\partial x_1} & \cdots & \frac{\partial q_n}{\partial x_1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \frac{\partial q_1}{\partial x_m} & & \cdots & \frac{\partial q_n}{\partial x_m} \end{bmatrix}$$

(12) Partial differentiation of the $n$ by 1 column matrix $q$ by a 1 by $m$ row matrix $x^T$ is the $n$ by $m$ matrix of partial derivatives with the element $\partial q_i/\partial x_j$ in the $i$th row, $j$th column. Thus with the comma convention we have

$$q_{,x^T} \triangleq \frac{\partial q}{\partial x^T} \triangleq \begin{bmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} & \cdots & \frac{\partial q_1}{\partial x_m} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \frac{\partial q_n}{\partial x_1} & & \cdots & \frac{\partial q_n}{\partial x_m} \end{bmatrix}$$

Note that $[q_{,x^T}]^T \equiv [q^T_{,x}]$ or $q_{,x^T}{}^T \equiv q^T_{,x}$.

(13) Partial differentiation of any $n$ by $m$ matrix, say $A$, with respect to any scalar, say $t$, is the $n$ by $m$ matrix of partial derivatives with the element $\partial A_{ij}/\partial t$ in the $i$th row, $j$th column. Thus

$$\frac{\partial A}{\partial t} \triangleq A_{,t} \triangleq [\partial A_{ij}/\partial t] \triangleq [A_{ij,t}]$$

(14) If $\omega$ is a 3 by 1 column matrix such that

$$\omega \triangleq \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

then $\tilde{\omega}$ is a 3 by 3 skew symmetric matrix given by

$$\tilde{\omega} \triangleq \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

# Appendix B

## Definition of Symbols

| Symbol | Page (equation) of first occurrence | Brief definition (see text for elaboration) |
|---|---|---|
| $A$ | 23 (55) | $m \times \nu$ matrix in constraint equation |
| $A_{sk}$ | 23 (54) | element of $A$ in row $s$, column $k$ |
| $A_1, A_2$ | 98 | partitions of $A$ |
| $A^k$ | 108 (307) | column matrix in HMH equations |
| $A^{mm}$ | 23 (56) | $m \times m$ partition of $A$ |
| $A^{mn}$ | 23 (56) | $m \times n$ partition of $A$ |
| $A'$ | 75 (219c) | partition of $A$ in sample problem |
| $\mathbf{A}$ | 121 | inertial acceleration of a particle or a system mass center |
| $a$ | 73 (215) | sphere radius in sample problem |
| $a_{ij}$ | 108 (307) | coefficient matrix in HMH equations |
| $B$ | 23 (55) | $m \times 1$ matrix in constraint equation |
| $B_s$ | 23 (54) | element of $B$ |
| $\mathscr{B}$ | 9 (24); 109 | number of rigid bodies; also set of indices corresponding to rigid bodies in a system |
| $\mathscr{B}$ | 16 (42) | number of deformable bodies |
| $b$ | 37 | rigid body |
| $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ | 37 (100) | dextral, orthogonal set of unit vectors fixed in $b$ |
| $\mathbf{b}^j_1, \mathbf{b}^j_2, \mathbf{b}^j_3$ | 88 | dextral, orthogonal unit vectors fixed in rigid body $\mathscr{b}_j$ |
| $\mathscr{b}_j$ | 88 | rigid body $j$ |
| $C^{ij}$ | 108 (307) | direction cosine matrix relating orientations of $\mathscr{b}_i$ and $\mathscr{b}_j$ |
| $c, CM$ | 15 | mass center |
| $c_j$ | 9 | mass center of body $j$ |
| $c_\alpha$ | 58 (170d) | $\cos \theta_\alpha$ |
| $\mathbf{c}_1$ | 88 (261) | position vector in sample problem |
| $D$ | 43 (117) | symmetric coefficient matrix of $q$; damping matrix |
| $\mathbf{D}^{sj}$ | 110 (309) | barycentric position vectors |
| $F$ | 39 (104) | right side of state equation |
| $F_j$ | 58 (169) | scalar component of $\mathbf{F}$ |
| $\mathbf{F}$ | 15, 34 (85) | resultant force |
| $\mathbf{F}_j$ | 4 (1) | force applied to particle $j$ |

# Definition of Symbols (contd)

| Symbol | Page (equation) of first occurrence | Brief definition (see text for elaboration) |
|---|---|---|
| $\mathbf{F}_j^e$ | 4 | "external force" applied to particle $j$ |
| $\mathbf{F}_j^i$ | 4 | "internal force" applied to particle $j$ |
| $\mathbf{F}^j$ | 89 (266a) | resultant force applied to body $j$ |
| $f$ | 12 (29) | floating reference frame with respect to which deformations are defined |
| $f, f_k$ | 28, 27, 29 | matrix of generalized active forces; $k$th element of $f$ |
| $f_T; f_B; f_P$ | 17 | Tisserand frame; Buckens frame; principal axis frame |
| $f^*, f_k^*$ | 28, 27, 29 | matrix of generalized inertial forces; $k$th element of $f^*$ |
| $\mathbf{f}$ | 9 (22) | resultant force on a rigid body, excluding "nonworking" constraint forces |
| $\mathbf{f}_j$ | 6 (13); 12 | force $\mathbf{F}_j$ minus nonworking constraint force $\mathbf{f}'_j$; also unit vector fixed in frame $f$ (for $j = 1, 2, 3$) |
| $\mathbf{f}'_j$ | 6 (12) | "nonworking" constraint force on particle $j$ |
| $\mathbf{f}^j$ | 9 (24) | force $\mathbf{F}^j$ minus nonworking constraint force |
| $G$ | 41 (109) | skew-symmetric matrix coefficient of generalized velocity matrix |
| $\mathbf{g}^k, g^k$ | 109 | unit vector defining hinge axis on $\mathscr{s}_k$ |
| $\mathbf{H}, H$ | 15 (39); 26 | angular momentum referred to system mass center; $\lvert\mathbf{H}\rvert$ |
| $\mathbf{H}^j$ | 10 (26) | angular momentum of body $j$ referred to its mass center |
| $h_i$ | 5 (9) | function appearing in holonomic constraint equation |
| $\mathscr{H}$ | 53 (156a) | Hamiltonian, expressed in terms of $q$, $p$, and $t$ |
| $I, I$ | 18,100 | inertia matrices |
| $I_{ij}$ | 101 | element of inertia matrix $I$ $(i, j = 1, 2, 3)$ |
| $I_s$ | 78 (234) | moment of inertia of uniform sphere for mass center |
| $I_1, I_2, I_3$ | 58 (169) | principal axis moments of inertia for body mass center (superscript may identify body index) |
| $I', I''$ | 79 (238), 83 (251) | coefficient matrices in sample problem |

## Definition of Symbols (contd)

| Symbol | Page (equation) of first occurrence | Brief definition (see text for elaboration) |
|---|---|---|
| $\mathbf{I}$ | 17 (43), 26 | inertia dyadic for mass center (superscript may identify body index) |
| $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ | 20 | inertially fixed, dextral, orthogonal unit vectors |
| $J^{\alpha}$ | 41 (109) | $n \times n$ matrix in equations of motion |
| $\mathbf{J}^{f}; J$ | 31 (72); 87 (259) | inertia dyadic referred to point $p$; moment of inertia in sample problem |
| $K$ | 42 (116) | coefficient matrix of $q$ in linearized equations |
| $k$ | 87 (259); 42 (116b) | spring constant in sample problem; also intermediate coefficient matrix for $q$ |
| $L$ | 88 (260a) | length in sample problem |
| $\mathbf{L}^{sj}$ | 109 (308) | position vectors of hinge points relative to mass centers |
| $\mathcal{L}$ | 30 | Lagrangian |
| $M, M'$ | 84 (253a), 40 (106a), 70 (210e) | inertia matrices |
| $\mathbf{M}; M_j$ | 34 (86); 58 (169) | resultant moment referred to mass center $c$; scalar component of $\mathbf{M}$ |
| $\mathcal{M}$ | 14 (36) | system mass |
| $\mathcal{M}_j$ | 10 | mass of rigid body $j$ |
| $m$ | 5 (9) | number of constraint equations |
| $m_j$ | 4 (1) | mass of particle $j$ |
| $\mathbf{m}^{j}$ | 10 (25) | moment about mass center $c_j$ of forces other than nonworking constraint forces applied to body $j$ |
| $\mathbf{m}^{p}$ | 9 (22) | moment for point $p$ of forces except for nonworking constraint forces |
| $\mathbf{m}^{p_j}$ | 9 (24) | moment about $p_j$ fixed in body $j$ of forces applied to body $j$, excluding nonworking constraint forces |
| $N$ | 4 (1) | number of particles |
| $N_j$ | 10 | number of particles in rigid body system $j$ |
| $n$ | 1; 6 | dimension of column matrix; also number of independent generalized coordinates (degrees of freedom) |
| $n$ | 12 (31) | number of modes of deformation |
| $O$ | 98 | point occupied by mass center in nominal state |

# Definition of Symbols (contd)

| Symbol | Page (equation) of first occurrence | Brief definition (see text for elaboration) |
|---|---|---|
| $P$ | 39 (104); 74 (218) | $2n \times 2n$ coefficient matrix in state equation; also matrix in sample problem |
| $p$ | 8 | reference point, often fixed in a rigid body |
| $p_k$; $p$ | 52 (152); 52 (153) | generalized momentum; matrix of generalized momenta |
| $\mathbf{p}$ | 98 (286) | generic position vector referred to $O$ |
| $\mathbf{p}_j$ | 18 | unit vector fixed in the principal axis frame |
| $\mathcal{P}$ | 9 (24); 109 | number of particles; also set of indices 1 through $n$ |
| $Q$ | 8 (16) | matrix of generalized active forces |
| $Q$ | 52 (149) | nonconservative part of generalized active force |
| $Q_k$ | 7 (14a) | generalized active force for coordinate $k$ |
| $Q_k'$ | 44 (122) | generalized constraint force for coordinate $k$ |
| $Q_k^*$ | 7 (14b) | generalized inertia force for coordinate $k$ |
| $Q^1, Q^0$ | 98 | partitions of $Q$ |
| $Q^*$ | 8 (16) | matrix of generalized inertia forces |
| $q$ | 23 | column matrix with typical element $q_j$ |
| $q$ | 23 | $n \times 1$ matrix partition of $q$ (unconstrained coordinates) |
| $q_j$ | 5 (6) | generalized coordinate $j$ |
| $q^c$ | 23 | $m \times 1$ matrix partition of $q$ (constrained coordinates) |
| $q^1, q^0$ | 98 | partitions of $q$ |
| $R$ | 88 (260a) | mass-weighted length in sample problem |
| $R_j$ | 58 (169) | scalar component of $\mathbf{R}$ |
| $\mathbf{R}$ | 8 (17) | generic symbol for inertial position vector |
| $\mathbf{R}_j$ | 4 (1) | inertial position vector for particle $j$ |
| $\mathbf{R}^c$ | 15 | inertial position vector of mass center $c$ |
| $\mathbf{R}^j$ | 10 (25) | inertial position vector of mass center $c_j$ of body $j$ |
| $\mathbf{R}^p$ | 8 | inertial position vector of reference point $p$ |
| $\mathbf{R}^{p_j}$ | 9 (24) | inertial position vector of reference point $p_j$ fixed in rigid body $j$ |
| $\mathcal{R}$ | 43 | Rayleigh damping function |
| $r$ | 87 (259) | dimension in sample problem |

## Definition of Symbols (contd)

| Symbol | Page (equation) of first occurrence | Brief definition (see text for elaboration) |
|---|---|---|
| $r_j$ | 12 (30) | position coordinates, scalar components of $\mathbf{r}$ |
| $\mathbf{r}$ | 12 (29) | position vector relative to $p$ of point in undeformed body |
| $\mathbf{r}_j$ | 8 | position vector of particle $j$ relative to $p$ |
| $\mathbf{r}^c$ | 31 (73) | position vector of mass center $c$ relative to reference point $p$ |
| $s_\alpha$ | 58 (170d) | $\sin \theta_\alpha$ |
| $T$ | 31 (71); 29 (68) | kinetic energy expressed in primitive terms or in terms of $q$, $q$, and $t$ |
| $T$ | 48 (137), 68 (203) | kinetic energy expressed in terms of $q$, $u$, and $t$, or in terms of $q$, $p$, and $t$ |
| $T_r$ | 35 | rotational kinetic energy |
| $T_t$ | 35 | translational kinetic energy |
| $T_2, T_1, T_0$ | 39 (105) | parts of $T$ that contain generalized velocity terms of degree 2, 1, and 0, respectively |
| $\mathbf{T}_j$ | 19 | unit vectors (for $j = 1, 2, 3$) fixed in the Tisserand frame |
| $\mathbf{T}^j$ | 109 (308) | resultant external torque applied to $\mathscr{B}_j$ for $c_j$ |
| $t$ | 5 (6) | time |
| $U$ | 27 | unit matrix |
| $U^j$ | 18 | $3 \times 1$ matrices (for $j = 1, 2, 3$) representing unit vectors |
| $\mathsf{U}$ | 17 | unit dyadic |
| $u_j$ | 17 (45); 26 (64a) | scalar components of $\mathbf{u}$ ($j = 1, 2, 3$); also derivative of quasi-coordinate $j$ |
| $\mathbf{u}$ | 12 (29) | deformational displacement |
| $V$ | 30 | potential energy |
| $\mathbf{V}_k$ | 25 (59b); 29 (65i) | coefficient of $q_k$, or $u_k$, in expression for $\mathbf{R}$ |
| $\mathbf{V}_t$ | 25 (59b); 29 (65i) | term in $\mathbf{R}$ independent of generalized velocities, or of quasi-coordinate derivatives |
| $\mathbf{V}_t^c$ | 25 (62a) | term in $\mathbf{R}^c$ independent of generalized velocities |
| $\mathbf{V}_k^c$ | 25 (62a) | coefficient of $q_k$ in expression for $\mathbf{R}^c$ |
| $\mathbf{V}_k^{c_j}$ | 26 (60c) | coefficient of $q_k$ in expression for $\mathbf{R}^j$ |
| $\mathbf{V}_k^j$ | 24 (59a) | coefficient of $q_k$ in expression for $\mathbf{R}_j$ |
| $\mathbf{V}_t^j$ | 24 (59a) | term in $\mathbf{R}_j$ independent of generalized velocities |

# Definition of Symbols (contd)

| Symbol | Page (equation) of first occurrence | Brief definition (see text for elaboration) |
|---|---|---|
| $W$ | 27 (64c) | coefficient matrix in quasi-coordinate derivative definition |
| $W_{sk}$ | 26 (64b) | element of $W$ in row $s$, column $k$ |
| $W_0$ | 97 (280) | $3 \times 3$ matrix partition of $W$ for special case |
| $w$ | 27 (64c) | column matrix in quasi-coordinate derivative definition |
| $w_k$ | 26 (64b) | element of $w$ |
| $x$ | 39 (103); 55 (166) | state matrix, consisting of either $q$ and $q$ partitions, or $q$ and $p$ partitions |
| $y$ | 99 (289) | integration variable in sample problem |
| $\Gamma$ | 40 (106b) | postmultiplier of $q^T$ in expression for $T_1$ |
| $\gamma$ | 50 (147b) | matrix appearing in Lagrange's quasi-coordinate equations |
| $\gamma_1, \gamma_2$ | 87 (259) | angles in sample problems |
| $\delta$ | 4 (4) | variational symbol |
| $\varepsilon_{ij}$ | 108 (307) | path element |
| $\eta_1, \eta_2$ | 88 (262) | modal coordinates in sample problem |
| $\theta_1, \theta_2, \theta_3$ | 58 (170) | 1-2-3 attitude angles |
| $\kappa$ | 42 (114) | $n \times n$ matrix representing approximation of $T_0$ |
| $\lambda$ | 46 (131) | matrix of Lagrange multipliers |
| $\lambda_s$ | 45 (126) | Lagrange multiplier |
| $\nu$ | 5 | number of (possibly interdependent) generalized coordinates |
| $\boldsymbol{\rho}$ | 15 | position vector of generic point referred to system mass center |
| $\boldsymbol{\rho}$ | 15 | position vector of generic point of undeformed system for system mass center |
| $\boldsymbol{\rho}_s$ | 10 (27), 109 | position vector of particle $s$ or mass center of $\mathcal{B}_s$ referred to system mass center |
| $\rho_j$ | 17 (45) | scalar components of $\boldsymbol{\rho}$ ($j = 1, 2, 3$) |
| $\tau_j$ | 108 (307) | magnitude of hinge torque |
| $\boldsymbol{\varphi}^j$ | 12 (31) | vector deformation functions of spatial variables, or modal vectors |
| $\omega_j$ | 58 (169) | scalar component of $\boldsymbol{\omega}$ |
| $\boldsymbol{\omega}$ | 8 | inertial angular velocity of a rigid body |

# Definition of Symbols (contd)

| Symbol | Page (equation) of first occurrence | Brief definition (see text for elaboration) |
|---|---|---|
| $\boldsymbol{\omega}^j, \omega^j$ | 9 (24), 108 (307) | inertial angular velocity of body $j$; matrix counterpart |
| $\boldsymbol{\omega}_k$ | 25 (62b) | coefficient of $q_k$ in expression for $\boldsymbol{\omega}$ |
| $\boldsymbol{\omega}_t$ | 25 (62b) | term in $\boldsymbol{\omega}$ independent of generalized velocities |
| $\boldsymbol{\omega}_k^j$ | 28 (65f) | coefficient of $u_k$ in expression for $\boldsymbol{\omega}^j$ |
| $\boldsymbol{\omega}_t^j$ | 28 (65f) | term in $\boldsymbol{\omega}^j$ independent of quasi-coordinate derivatives |



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